Higher Algebra – MATH 570

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Contents

1. Introduction

We already assume some familiarity with modules, groups, fields, ...

2. Categories

2.1. **Categories: definition and some examples.** The notion of a category serves to organize the type of objects we study in mathematics and the maps between the objects that we allow. But besides serving as a way to systemize our conventions, it also introduces powerful tools, namely, functors and derived functors, equivalence of categories, universal objects, adjoint functors and so on, that allow one to prove new results and clarify the wheels at work behind many results in algebra, topology and logic.

2.1.1. *Definition of category.* A category $\mathbf{K}$ consists of the following data:

(1) A collection of objects Ob $\mathbf{K}$ of objects of $\mathbf{K}$;
(2) For any two objects $A, B \in \mathrm{Ob}\ \mathbf{K}$ a set $\mathrm{Mor}(A, B)$, called the morphisms from A to B, such that the following hold:
   (a) The sets $\mathrm{Mor}(A, B)$ are pairwise disjoint;
   (b) There is an associative composition law
   $$\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C), \qquad (f, g) \mapsto g \circ f,$$
   (often denoted gf);
   (c) For every object A there is a morphism $1_A \in \mathrm{Mor}(A, A)$ such that $f 1_A = f, f \in \mathrm{Mor}(A, B)$ and $1_A f = f, f \in \mathrm{Mor}(B, A)$.

(It follows that 1_A is unique.) A morphism $f \in \mathrm{Mor}(A, B)$ is called an isomorphism, or equivalence, if there is a morphism $g \in \mathrm{Mor}(B, A)$ such that $gf = 1_A, fg = 1_B$. Note that such g, if it exists, is unique since if also $g'f = 1_A$ then $(g'f)g = g$ but $(g'f)g = g'(fg) = g'1_B = g'$. We can therefore denote g by f^{-1} when convenient.

2.1.2. *Examples.* Here we list some of the principal examples of categories. Some may be unfamiliar at this point, but will make sense when you continue in your studies of mathematics. The verification of the axioms is left to you.

(1) The category of sets **Set**. The objects are sets. (Note that this collection is not a set in itself; the notion of cardinality doesn't make sense for it, since if we denote this collection by S then the cardinality of 2^S is bigger than that of S, by a fundamental result, yet $2^S \subset S$ because every element of 2^S is a subset of S, hence a set in its own right, thus an element of S.) The morphisms are just functions.
(2) The category of partially ordered sets, or posets, **POSet**. The objects are sets $(S, \leq)$ together with a partial order $\leq$ (a relation which is reflexive, transitive and $x \leq y, y \leq x \Rightarrow x = y$, but there could be elements x, y of S for which neither $x \leq y$, nor $y \leq x$.) The morphisms are functions that preserve order, namely $f \in \mathrm{Mor}((A, \leq), (B, \leq))$ if f is a function from A to B and $x \leq y, x, y \in A$ implies $f(x) \leq f(y)$.
(3) The category of linearly ordered sets, **LOSet**. The objects are posets $(A, \leq)$ only that the order is complete: for $x, y \in A$ either $x \leq y$ or $y \leq x$. The morphisms are as for **POSet**.
(4) The category of groups **Gp**. The objects are groups and the morphisms are group homomorphisms.
(5) The category of abelian groups **AbGp**. The objects are abelian groups and the morphisms are group homomorphisms.
(6) The category of left R-modules ${}_{\mathrm{R}}\mathbf{Mod}$ over a ring R. Let R be a ring, always associative with identity 1. The objects are left R-modules (so the structure map is $R \times M \to M$). The morphisms

are R-modules homomorphisms. Similarly we have the category of right R-modules $\mathbf{Mod}_{\mathrm{R}}$ (so the structure map is $M \times R \to M$). In particular, if R is a field then ${}_{\mathrm{R}}\mathbf{Mod}$ is the category of vector spaces over R whose morphisms are linear maps.

(7) The category of rings **Ring**. The objects are rings and the morphisms are ring homomorphisms.

(8) The category of topological spaces **TopSp**. The objects are topological spaces and the morphisms are continuous maps.

(9) The category of short exact sequences of left R-modules, ${}_{\mathrm{R}}\mathbf{SES}$. Let R be a ring. Recall that a short exact sequence of R-modules is a sequence of left R-modules

$$0 \longrightarrow M_1 \xrightarrow{i} M_2 \xrightarrow{j} M_3 \longrightarrow 0\ ,$$

such that the image of every map is the kernel of the following one, that is, i is injective, $\mathrm{Im}(i) = \mathrm{Ker}(j)$ and j is surjective. A morphism,

$$(\ 0 \longrightarrow M_1 \xrightarrow{i} M_2 \xrightarrow{j} M_3 \longrightarrow 0\ ) \to (\ 0 \longrightarrow N_1 \xrightarrow{i'} N_2 \xrightarrow{j'} N_3 \longrightarrow 0\ ),$$

is a triple (f_1, f_2, f_3) where $f_i : M_i \to N_i$ is a morphism of R-modules and the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "i"] \arrow[d, "f_1"] & M_2 \arrow[r, "j"] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "i'"] & N_2 \arrow[r, "j'"] & N_3 \arrow[r] & 0.
\end{tikzcd}$$

(10) Let S be a monoid (a semi-group with a two sided inverse). That is S has an associative composition law, $(x, y) \mapsto xy$, with a two-sided inverse. Define a category $\mathbf{C}$, with a single object $*$ and with $\mathrm{Mor}(*, *) = S$, where composition is given by multiplication. (So morphisms need not be functions.)

(11) The category of linear representations of a finite group G over a field k, $\mathbf{Rep}_{\mathrm{k}}(\mathbf{G})$. Let G be a finite group and k a field. A linear representation of G over k is a group homomorphisms $\rho : G \to \mathrm{GL}(V)$, where V is a vector space over k. The objects of the category $\mathbf{Rep}_{\mathrm{k}}(\mathbf{G})$ are finite dimensional linear representations of G over k. A morphism $f : (V, \rho) \to (V', \rho')$ is a linear map $f : V \to V'$ such that $f \circ \rho(g) = \rho'(g) \circ f$ for all $g \in G$. Namely, f is a G-equivariant linear map. We remark that the finiteness of G plays no role so far. However, having in mind the theory we are about to develop concerning representations of groups, we restrict our discussion from the outset to finite groups.

(12) Let G be a group. The category **GSet** is the category whose objects are pairs (S, ρ) consisting of a non-empty set S and a group homomorphism $\rho : G \to \Sigma_S$ from G to the group of permutations of S; namely, sets S equipped with a G-action. The morphisms are G-equivariant functions.

Exercise 2.1.1. An object X in a category $\mathbf{C}$ is called a final (respectively, initial) object, if for every object A in $\mathbf{C}$ there is a unique morphism $A \to X$ (respectively, $X \to A$).

(1) Prove that an initial (resp. final) object is unique, up to unique isomorphism, if it exists.
(2) For each category listed above find if there is a final, or initial, object and determine it.

2.2. **Functors and natural transformations.**

2.2.1. *Definition of a functor.* Let $\mathbf{K}, \mathbf{H}$ be two categories. A covariant functor $F : \mathbf{K} \to \mathbf{H}$ from $\mathbf{K}$ to $\mathbf{H}$ is a rule associating to every object A of $\mathbf{K}$ an object $F(A)$ of $\mathbf{H}$ and to every morphism $f \in \mathrm{Mor}(A, B)$ a morphism $F(f) \in \mathrm{Mor}(F(A), F(B))$ such that (i) composition is respected, $F(gf) = F(g)F(f)$, and (ii) $F(1_A) = 1_{F(A)}$.

We remark that not every object of $\mathbf{H}$ needs to be of the form $F(A)$ for some object A; likewise, not every morphism $h \in \mathrm{Mor}(F(A), F(B))$ needs to be of the form $F(f)$ for some $f \in \mathrm{Mor}(A, B)$. It is also possible that for some $f_1 \neq f_2 \in \mathrm{Mor}(A, B)$ we have $F(f_1) = F(f_2)$. This motivates the following definitions: (i) If for every A, B in $\mathbf{Ob\,K}$, for $f_1, f_2 \in \mathrm{Mor}(A, B)$ the equality $F(f_1) = F(f_2)$ implies $f_1 = f_2$, we call F faithful; (ii) If every morphism $h \in \mathrm{Mor}(F(A), F(B))$ is the form $F(f)$ for some $f \in \mathrm{Mor}(A, B)$, and that holds for all A, B in $\mathbf{Ob\,K}$, we call F full.

A contravariant functor $F : \mathbf{K} \to \mathbf{H}$ from $\mathbf{K}$ to $\mathbf{H}$ is a rule associating to every object A of $\mathbf{K}$ an object $F(A)$ of $\mathbf{H}$ and to every morphism $f \in \mathrm{Mor}(A, B)$ a morphism $F(f) \in \mathrm{Mor}(F(B), F(A))$ such that (i) composition is respected, $F(gf) = F(f)F(g)$, and (ii) $F(1_A) = 1_{F(A)}$.

2.2.2. *Examples.* We shall see many examples during the course. For now we give some basic, sometimes artificial, examples.

(1) There is a class of functors, called forgetful functors. They are often faithful and rarely full. Here are some instances:
   (a) $F : \mathbf{POSet} \to \mathbf{Set}$, $F((A, \leq)) = A$, $F(f) = f$. The functor forgets the order on A and forgets the fact that f preserves order. This functor is faithful, but not full.
   (b) $F : \mathbf{LOSet} \to \mathbf{POSet}$, $F((A, \leq)) = (A, \leq)$, $F(f) = f$. The functor forgets the fact that the order is complete. This functor is full and faithful.
   (c) $F : \mathbf{Rep}_{\mathrm{k}}(\mathbf{G}) \to {}_{\mathrm{k}}\mathbf{Mod}$. Here $F((V, \rho)) = V, F(f) = f$. The functor forgets the group action and the fact that f is equivariant.
   (d) $F : \mathbf{TopSp} \to \mathbf{Set}$, $F((S, \mathscr{T})) = S$, $F(f) = f$. The functor forgets that the set S has a topology $\mathscr{T}$ on it and forgets the fact that f is continuous.

(2) The functor $F : \mathbf{Gp} \to \mathbf{AbGp}$. Let G be a group. Recall that G' denotes the commutator group of G, the subgroup of G generated by all expressions $[x, y], x, y \in G$, where $[x, y] = xyx^{-1}y^{-1}$. The group G' is normal in G and G/G' is the largest quotient of G which is abelian. The functor is $F(G) = G/G'$ and $F(f) = \bar{f}$, where by that we mean that $f : G \to H$ induces a well-defined group homomorphism, denoted $\bar{f}$, from G/G' to H/H', $\bar{f}(gG') = f(g)H'$.

   Exercise 2.2.1. Prove that this functor is not faithful and not full.

(3) The restriction and induction functors. Let H be a subgroup of a finite group G. The restriction functor $\mathrm{Res} : \mathbf{Rep}_{\mathrm{k}}(\mathbf{G}) \to \mathbf{Rep}_{\mathrm{k}}(\mathbf{H})$ is given by $\mathrm{Res}((V, \rho)) = (V, \rho|_H)$. If we need to be more precise we shall denote this functor by Res^G_H. For a morphism $f : (V, \rho) \to (V, \rho')$ we have $F(f) = f$. This functor is faithful but not full.

   There is also a functor, called induction, $\mathrm{Ind} : \mathbf{Rep}_{\mathrm{k}}(\mathbf{H}) \to \mathbf{Rep}_{\mathrm{k}}(\mathbf{G})$. Given a representation (W, α) of H, we let

$$V = \{f : G \to W : f(gh) = \alpha(h)^{-1}f(g), h \in H\},$$

which is k-vector space under the usual addition and multiplication by scalar of functions. The action of G is given by

$$[\rho(g)f](x) = f(g^{-1}x).$$

In the literature one also finds the following space

$$V' = \{f\colon G \to W : f(hg) = \alpha(h)f(g), \forall h \in H\},$$

with G acting by

$$[\rho'(g)f](x) = f(xg).$$

Exercise 2.2.2. Prove that V and V' are representations of G, and are in fact isomorphic representations. Show how to complete the definition so that we get a functor. Is the functor full? faithful?

(4) We have the torsion functor Tor from **AbGp** to **AbGp** associating to an abelian group A its torsion subgroup $\mathrm{Tor}(A) = \{a \in A : o(a) < \infty\}$. Here $\mathrm{Tor}(f) = f$. This functor is not faithful. If A is finitely generated then $\mathrm{Hom}(A, B) \to \mathrm{Hom}(\mathrm{Tor}(A), \mathrm{Tor}(B))$ is surjective, but for general abelian groups seems it is not, and so this is not a full functor. Suppose that there is an abelian group G such that there is no surjection $G \to \mathrm{Tor}(G)$ extending the identity map on $\mathrm{Tor}(G)$. Take the identity map $\mathrm{Tor}(G) \to \mathrm{Tor}(G)$. If there is a homomorphism $f : G \to \mathrm{Tor}(G)$ extending the identity map then f is surjective and we get a contradiction and so Tor applied to $\mathrm{Hom}(G, \mathrm{Tor}(G))$ is not surjective onto $\mathrm{Hom}(\mathrm{Tor}(G), \mathrm{Tor}(G))$. Conversely, if for every G there is a surjection $G \to \mathrm{Tor}(G)$ extending the identity map on $\mathrm{Tor}(G)$ then we easily check that Tor is full (given $f : \mathrm{Tor}(G) \to \mathrm{Tor}(H)$ take $G \twoheadrightarrow \mathrm{Tor}(G) \to \mathrm{Tor}(H) \hookrightarrow H$). Thus, the issue is whether there is an abelian group G for which $\mathrm{Tor}(G)$ is not a direct summand? It is known that if $\mathrm{Tor}(G)$ is killed by some positive integer N then it is a direct summand of G (see Rotman: An introduction to the theory of groups, 3rd edition, Corollary 10.37), but it is not so in general:

Consider the direct product $A = \prod_{n=0}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$ and let $T = \mathrm{Tor}(A)$ be its torsion subgroup. Now, suppose that T is a direct summand, say $A = T \oplus C$. On the one hand, in $A/T \cong C$ the non-zero element $(0, 0, p, p, p^2, p^2, \dots)$ is divisible by every power of p. Indeed, given $n \geq 0$, up to torsion, this element is equal to $p^n \cdot (0, \dots, 0, \underset{2n}{1}, 1, p, p, p^2, p^2, \dots)$. On the other, in A and so in C, for every element x there is a maximal natural number n such that x is divisible by p^n. In fact, take the first non-zero coordinate of x, say $x_i \in \mathbb{Z}/p^i\mathbb{Z}$. Then n is the maximal integer a such that $x_i \equiv 0 \pmod{p^a}$. We get a contradiction.

(5) Contravariant functors appear frequently, often in the context of duality, or in the context of cohomology. Here is a simple example. Let k be a field and consider on the category ${}_{\mathrm{k}}\mathbf{Mod}$ the functor $*$ associating to a vector space V the dual vector space $V^* = \mathrm{Hom}_k(V, k)$ and to a map $f : V \to W$ the map $f^* : W^* \to V^*, f^*(\phi)(v) = \phi(f(v))$.

Exercise 2.2.3. Let $\mathbf{f}_{\mathrm{k}}\mathbf{Mod}$ be the category of finitely generated k-modules (i.e., finite dimensional k-vector spaces). Prove that $*$ is also a functor on $\mathbf{f}_{\mathrm{k}}\mathbf{Mod}$, and that it is full and faithful. Is this functor full or faithful on **kMod**?

(6) Another example of contravariant functors is the following. Let L/K be a finite Galois extension of fields, namely the group of field automorphisms of L that fix K, $\mathrm{Aut}_K(L)$ has cardinality $[L : K]$

(and is denoted in this case $\mathrm{Gal}(L/K)$). Let the category $\mathbf{K}$ consist of the subfields $L \supseteq L' \supseteq K$ and $\mathrm{Mor}(L', L'')$ consists only of the inclusion map $L' \hookrightarrow L''$ if L' is a subfield of L'' and otherwise is the empty set. Let G be the Galois group $\mathrm{Gal}(L/K)$ and let $\mathbf{C}$ be the category whose objects are subgroups H of $\mathrm{Gal}(L/K)$ with, again, morphism being the inclusion maps only, if they exist.

The Galois correspondence $L' \mapsto G^{L'} = \{g \in G : g|_{L'} = id\}$ is a contravariant functor $\mathbf{K} \to \mathbf{C}$, and the correspondence $H \mapsto L^H = \{\ell \in L : h(\ell) = \ell, \forall h \in H\}$ is a contravariant functor $\mathbf{C} \to \mathbf{K}$. The main theorem of Galois theory is that these functors are inverses to each other.

2.2.3. *Definition of a natural transformation.* Let $F, G : \mathbf{K} \to \mathbf{H}$ be two functors. A natural transformation α from F to G is a map associating to every object A in $\mathbf{Ob\,K}$ a morphism $\alpha_A : F(A) \to G(A)$ such that for every arrow $f : A \to B$ in $\mathbf{K}$ we have a commutative diagram:

$$\begin{tikzcd} A \arrow[d, "f"] & F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"] & G(A) \arrow[d, "G(f)"] \\ B & F(B) \arrow[r, "\alpha_B"] & G(B). \end{tikzcd}$$

If each α_A is an isomorphism, we say that F and G are naturally equivalent, or isomorphic.

We only give a few examples at this point. Given a set S there are two trivial topologies on it: the topology $\mathscr{T}_{\mathrm{disc}}$ consisting of all subsets of S, and the topology $\mathscr{T}_{\mathrm{triv}}$ consisting of the empty set and the total space alone. We get two functors $F, G : \mathbf{Set} \to \mathbf{TopSp}$, $F(S) = (S, \mathscr{T}_{\mathrm{disc}}), F(f) = f$ and $G(S) = (S, \mathscr{T}_{\mathrm{triv}}), G(f) = f$. There is natural transformation $\alpha : F \to G$ given by $\alpha_A = 1_A$. Note, though, that there is no natural transformation $G \to F$.

Here is a another example. Consider the abelianization functor $G \mapsto G/G'$, now as a functor from $\mathbf{Gp}$ to $\mathbf{Gp}$ (and not to $\mathbf{AbGp}$). The natural homomorphism $\alpha_G : G \to G/G'$ defines a natural transformation from the identity functor to the abelianization functor. There is no natural transformation in the other direction.

As a final example, consider the double dual functor $V \mapsto V^{**}$ on the category of k-vector spaces $\mathbf{kMod}$. The natural map $V \to V^{**}$, mapping a vector v to the function sending a functional ϕ on V to $\phi(v)$, defines a natural transformation of the identity functor to the functor $**$.

We have a similar definition of a natural transformation between two contravariant functors.

Let $F, G : \mathbf{K} \to \mathbf{H}$ be two contravariant functors. A natural transformation α from F to G is a map associating to every object A in $\mathbf{Ob\,K}$ a morphism $\alpha_A : F(A) \to G(A)$ such that for every arrow $f : A \to B$ in $\mathbf{K}$ we have a commutative diagram:

$$\begin{tikzcd} A \arrow[d, "f"] & F(A) \arrow[r, "\alpha_A"] & G(A) \\ B & F(B) \arrow[r, "\alpha_B"] \arrow[u, "F(f)"] & G(B). \arrow[u, "G(f)"'] \end{tikzcd}$$

If each α_A is an isomorphism, we say that F and G are naturally equivalent, or isomorphic.

2.3. **Equivalence of categories.**

2.3.1. *Definition of equivalence.* Let **K** and **C** be categories. We say they are equivalent if there are functors $F : \mathbf{K} \to \mathbf{C}$ and $G : \mathbf{C} \to \mathbf{K}$ such that the composition $GF \cong 1_{\mathrm{K}}$ (the identity functor of **K**) and $FG \cong 1_{\mathrm{C}}$ (the identity functor of **C**).

We have similarly the notion of antiequivalence. The definition is the same only that both F and G are assumed to be contravariant (note that the compositions are covariant so the requirements $GF \cong 1_{\mathrm{K}}$, $FG \cong 1_{\mathrm{C}}$, make sense).

2.3.2. *Some examples.* Here are some important examples.

(1) The categories of subfields of a Galois extension and subgroups of the Galois group, cf. § 2.2.2 are antiequivalent.

(2) The functor $*$ on the category of k-vector spaces **kMod** is not an antiequivalence, in general we only have a natural transformation $1 \mapsto **$ which is not an equivalence. The problem being that for infinite dimensional vector spaces the map $V \to V^{**}$ is only an inclusion.

Let **K** be a category. A subcategory **C** of **K** is a category whose objects are a subcollection of those of **K** and such that for every $A, B \in \mathbf{Ob}\,\mathbf{C}$ we have $\mathrm{Mor}_{\mathrm{C}}(A, B) \subseteq \mathrm{Mor}_{\mathrm{K}}(A, B)$. For example, the category of finite sets is a subcategory of the category of sets. A subcategory is called full if in fact we have $\mathrm{Mor}_{\mathrm{C}}(A, B) = \mathrm{Mor}_{\mathrm{K}}(A, B)$ for any $A, B \in \mathbf{Ob}\,\mathbf{C}$. Thus, the category of finite sets if a full subcategory of the category of sets. The category of abelian groups is a full subcategory of the category of groups. The category of finite dimensional vector spaces over k, $\mathbf{f_k Mod}$ is a full subcategory of the category ${}_kMod$. On the category $\mathbf{f_k Mod}$ the duality functor $*$ is an anti-equivalence.

Consider now another category, say **B**. The objects of **B** are the vector spaces $\{0\} = k^0, k = k^1, k^2, k^3, \dots$, one for each non-negative integer. The morphisms are just linear maps. There is an obvious functor $F : \mathbf{B} \to \mathbf{f_k Mod}$. Define a functor $G : \mathbf{f_k Mod} \to \mathbf{B}$. Given an object A in $\mathbf{f_k Mod}$, choose an isomorphism $\eta_A : A \to k^{\dim(A)}$; if $A = k^n$ then we choose η_A to be the identity. Define now $G(A) = k^{\dim(A)}$ and for $f \in \mathrm{Mor}(A, B)$ the map $G(f) = \eta_B f \eta_A^{-1}$.

Exercise 2.3.1. Prove that G is a functor and that (F, G) is a natural equivalence of categories. Note that though in a certain sense F and G are inverses, G is not uniquely determined by F; its definition very much depends on the choice of maps η_A.

2.3.3. *A criterion for equivalence.* There is a general criteria for a functor F to be a natural equivalence of categories. The proof is not harder than the example of $\mathbf{f_k Mod}$.

Theorem 2.3.2. *Let $F : \mathbf{C} \to \mathbf{D}$ be an functor. There exists a functor $G : \mathbf{D} \to \mathbf{C}$ such that (F, G) is an equivalence of categories if and only if (i) F is full and faithful; (ii) F is essentially surjective, namely, for every object D of $\mathbf{D}$ there is an object C of $\mathbf{C}$ such that $F(C)$ is isomorphic to D.*

Proof. Suppose that there exists such a functor G and let $\gamma : GF \to 1_{\mathrm{C}}$, $\delta : FG \to 1_{\mathrm{D}}$ be isomorphisms. For every $f : A \to B$ we have a commutative whose horizontal arrows are isomorphisms:

$$\begin{tikzcd} A \arrow[d, "f"] & GF(A) \arrow[r, "\gamma_A"] \arrow[d, "GF(f)"] & A \arrow[d, "f"] \\ B & GF(B) \arrow[r, "\gamma_B"] & B \end{tikzcd}$$

(and similarly for δ). Consider $\mathrm{Mor}_{\mathrm{C}}(A, B)$ and $\mathrm{Mor}_{\mathrm{D}}(F(A), F(B))$. It is easy to check that the isomorphism $\gamma : GF \to 1_{\mathrm{C}}$ induces an isomorphism

$$GF : \mathrm{Mor}(A, B) \cong \mathrm{Mor}(GF(A), GF(B)),$$

for every $A, B \in \mathbf{Ob\,C}$. Indeed, given $f : A \to B$ we get $GF(f) : GF(A) \to GF(B)$ and in fact $GF(f) = \gamma_B^{-1} f \gamma_A$, where $\gamma_A : GF(A) \to A$ etc. . Conversely, given $g : GF(A) \to GF(B)$ we have $\gamma_B g \gamma_A^{-1} : A \to B$. Since $GF : \mathrm{Mor}(A, B) \to \mathrm{Mor}(GF(A), GF(B))$ factors through $\mathrm{Mor}(A, B) \to \mathrm{Mor}(F(A), F(B))$, this map is injective and so F is faithful.

Likewise, the isomorphism $FG : \mathrm{Mor}(F(A), F(B)) \to \mathrm{Mor}(FGF(A), FG(B))$ factors through the map $F : \mathrm{Mor}(GF(A), GF(B)) \to \mathrm{Mor}(FGF(A), FGF(B))$ and so this map F is surjective. Since $GF \cong 1_{\mathrm{C}}$ we get $\mathrm{Mor}(A, B) \cong \mathrm{Mor}(GF(A), GF(B)) \to \mathrm{Mor}(FGF(A), FGF(B)) \cong \mathrm{Mor}(F(A), F(B))$ is surjective, too. This composition is calculated as follows: $f \mapsto \gamma_B^{-1} f \gamma_A \mapsto F(\gamma_B)^{-1} F(f) F(\gamma_A) \mapsto \delta_{F(B)} F(\gamma_B)^{-1} F(f) F(\gamma_A) \delta_{F(A)}^{-1}$. There is no reason for $F(\gamma_A)\delta_{F(A)}^{-1}$ to cancel out; nonetheless, $F(\gamma_A)\delta_{F(A)}^{-1}$ is an isomorphism and so is $\delta_{F(B)} F(\gamma_B)^{-1}$ and it follows that also the map $f \mapsto F(f)$ from $\mathrm{Mor}(A, B)$ to $\mathrm{Mor}(F(A), F(B))$ is surjective too. This shows that F must be full and faithful. Furthermore, let D be an object of $\mathbf{D}$ then $C = G(D)$ is an object of $\mathbf{C}$ and we have an isomorphism $\delta_D : FG(D) \to D$, and so the last condition is also satisfied.

Conversely, let F be a functor with the stated properties. To define G first choose in arbitrarily an isomorphism $\alpha_D : D \to F(C_D)$ for every object D in $\mathbf{D}$, where C_D is a suitable object of $\mathbf{C}$. Define G on objects by $G(D) = C_D$ and on morphisms as follows. Given a morphism $g \in \mathrm{Mor}_{\mathrm{D}}(D, E)$ we get a morphism $g' = \alpha_E g \alpha_D^{-1} \in \mathrm{Mor}(F(C_D), F(C_E))$. There is a unique morphism $f \in \mathrm{Mor}(C_D, D_E) = \mathrm{Mor}(G(D), G(E))$ such that $F(f) = g'$. We let $f = G(g)$.

To define $\delta : FG \to 1_{\mathrm{D}}$ we let $\delta_D : FG(D) = F(C_D) \to D$ be the morphism α_D^{-1}. To define $\gamma : GF \to 1_{\mathrm{C}}$ we let $\gamma_A : GF(A) = C_{F(A)} \to A$ be the unique morphism such that $F(\gamma_A) : F(C_{F(A)}) \to F(A)$ is $\alpha_{F(A)}^{-1}$. The verifications are left as an exercise. □

As an application we prove the following theorem, which is part of what's called "Morita equivalence".

Theorem 2.3.3. *Let R be a ring and $n \geq 1$ an integer. The categories ${}_R Mod$ and ${}_{M_n(R)} Mod$ are equivalent.*

Proof. We define a functor

$$F :_R Mod \to {}_{M_n(R)} Mod,$$

that satisfies the conditions of Theorem 2.3.2. Given an R-module M let

$$F(M) = M^n = \{{}^t(m_1, \dots, m_n) : m_i \in M\}.$$

It is naturally an $M_n(R)$-module. Given a morphism $f : M \to N$ we get a morphism $F(f) : M^n \to N^n$ by

$$F(f)(m_1, \dots, m_n) = (f(m_1), \dots, f(m_n)).$$

The verification that F is a functor is easy.

Let E_{ij} be the $n \times n$ elementary matrix with 1 in the ij entry and zeroes otherwise.

It is clear that F is faithful. Let $g : M^n \to M^n$ be a morphism of $M_n(R)$-modules. Then $g = (g_1, \dots g_n)$. Let $m \in M$.

We have $g(m, 0, \dots 0) = g(E_{11}{}^t(m, 0, \dots 0)) = E_{11}(g_1(m, 0, \dots 0), g_2(m, 0, \dots 0), \dots, g_n(m, 0, \dots 0)) = (g_1(m, 0, \dots 0), 0, \dots 0)$. This shows that $g_i(m, 0, dots, 0) = 0$. A similar argument thus gives that $g_i(me_j) =$

0 and so that $g_i(m_1, dots, m_n) = g_i(m_i)$. We conclude that $g(m_1, \dots, m_n) = (g_1(m_1), \dots g_n(m_n))$, where each $g_i : M \to M$ is an R-module map.

Now, given a permutation $\sigma \in S_n$, there is a matrix $M(\sigma) \in M_n(R)$ such that $M(\sigma)(m_1, \dots m_n) = (m_{\sigma(1)}, \dots, m_{\sigma(n)})$. We conclude that $g(M(\sigma)(m_1, \dots, m_n)) = (g_1(m_{\sigma(1)}), \dots, m_{\sigma(n)})$ and is equal also to $M(\sigma)(g_1(m_1), \dots g_n(m_n)) = (g_{\sigma(1)}(m_{\sigma(1)}), \dots g_{\sigma(n)}(m_{\sigma(n)}))$. We conclude that $g_i = g_{\sigma(i)}$ for all i and σ and so that $g_1 = \cdots = g_n$. Therefore $g = F(g_1)$ and we conclude that F is full.

Exercise 2.3.4. Prove that every $M_n(R)$-module is isomorphic to M^n for some R-module M.

□

2.4. **products and coproducts.** The notion of a product and coproduct (also called "sum") is very important in any category where it exists. It also allows us, in this exposition, to introduce the idea of a universal object.

Let $\mathbf{C}$ be a category and $A, B \in \mathbf{Ob\, C}$. The product of A, B, it it exists, is an object C together with two morphisms $\alpha : C \to A, \beta : C \to B$ such that for every object D of $\mathbf{C}$ and morphisms $\alpha' : D \to A, \beta' : D \to B$ there is a unique morphism $f : D \to C$ such that the following diagram commutes:

$$\begin{tikzcd}
& D \arrow[d, "f"] \arrow[ddl, bend right, "\alpha'"'] \arrow[ddr, bend left, "\beta'"] & \\
& C \arrow[dl, "\alpha"] \arrow[dr, "\beta"'] & \\
A & & B.
\end{tikzcd}$$

Remark 2.4.1. The use of arrows is meant to ease notation. Instead of writing $f \in \mathrm{Mor}(D, C)$ we write $f : D \to C$ and identities between compositions of morphisms are described by diagrams. There is no need to assume that a morphism is a function from a set D (with additional structure) to a set C (with additional structure). Though often this is the case, e.g for **Set**, **Gp**, **TopSp**, $_{\mathrm{R}}$**Mod**, it need not be so in general; already for **SES** the situation is more complicated.

Proposition 2.4.2. *The product of A, B, if it exists, is unique up to unique isomorphism. We denote it $A \prod B$.*

Proof. The proof is easy. Still, it is a prototype for many similar statements so we give it in detail. Suppose that C' is also a product with morphisms $\alpha' : C' \to A, \beta' : C' \to B$. Then we get a commutative diagram

$$\begin{tikzcd}
C' \arrow[r, shift left, "f"] \arrow[d, "\alpha'"] \arrow[dr, "\beta'" near start] & C \arrow[l, shift left, "f'"] \arrow[d, "\beta"] \arrow[dl, "\alpha" near end] \\
A & B.
\end{tikzcd}$$

From which we conclude a commutative diagram

$$\begin{tikzcd}
C \arrow[r, shift left, "ff'"] \arrow[r, shift right, "1_C"'] \arrow[d, "\alpha"] \arrow[dr, "\beta" near start] & C \arrow[d, "\beta"] \arrow[dl, "\alpha" near end] \\
A & B.
\end{tikzcd}$$

The uniqueness of the morphism $C \to C$ making the diagram commutative, implies $ff' = 1_C$. Similarly, $f'f = 1_{C'}$ and we conclude that C is isomorphic to C'. $\square$

The notion of coproduct is similar. Let $A, B \in \mathbf{Ob\,C}$. The product of A, B, it it exists, is an object C together with two morphisms $\alpha : A \to C, \beta : B \to C$ such that for every object D of $\mathbf{C}$ and morphisms $\alpha' : A \to D, \beta' : B \to D$ there is a unique morphism $f : C \to D$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[dr, "\alpha"] \arrow[ddr, bend right, "\alpha'"'] & & B \arrow[dl, "\beta"'] \arrow[ddl, bend left, "\beta'"] \\
& C \arrow[d, "f"] & \\
& D. &
\end{tikzcd}$$

Likewise, if the coproduct exists it is unique up to unique isomorphism. The similarity can be given an explanation, useful in many other cases. Given a category $\mathbf{C}$ define the dual category $\mathbf{C}^*$ to be the category with objects $\mathbf{Ob\,C}$ and morphisms $\mathrm{Mor}_{\mathbf{C}^*}(A, B) = \mathrm{Mor}_{\mathbf{C}}(B, A)$ with the natural composition law: $g \circ_{\mathbf{C}^*} f = f \circ_{\mathbf{C}} g$. Under this construction the notion of product and coproduct is interchanged, and the notion of isomorphism is preserved. Thus the uniqueness of the coproduct of a category $\mathbf{C}$ follows from the uniqueness of the product for the category $\mathbf{C}^*$, which was already established.

In general it can be rather tricky to determine when a product and coproduct exist; the arguments are specific to the case under consideration.

(1) **Set**. In this case the product is the cartesian product and the coproduct is the disjoint union.
(2) **TopSp**. In this case the product is the cartesian product, with the product topology, and the coproduct is the disjoint union with the disjoint union topology. There is a subtle point involving the choice of topologies in verifying the universal property.
(3) $_{\mathrm{R}}$**Mod**. Here both the product and the coproduct are both the cartesian product. If one defines infinite products and coproducts then the product is the (unrestricted) direct product, while the coproduct is the set of elements in the direct product all but finitely many of their coordinates are zero. Thus, the product of a family of R-modules $\{M_i : \imath \in I\}$, indexed by a set I, is

$$\prod_{i \in I} M_i = \{(m_i)_{i \in I} : m_i \in M_i\},$$

where addition is done component wise and R acts diagonally.

The direct sum, or coproduct, is usually denoted either $\coprod_{i \in I} M_i$, or $\oplus_{i \in I} M_i$, and consists of

$$\{(m_i)_{i \in I} : m_i \in M_i, m_i = 0 \text{ except for finitely many } i\}.$$

(4) **LOSet** and **POSet**.

Exercise 2.4.3. Prove the assertions concerning product and coproduct in **Set**, **TopSp** and $_{\mathrm{R}}$**Mod**. Prove that in **LOSet** products and coproducts need not exist, but that they exist in **POSet**.

(5) **Gp**. The product is simply the direct product. The coproduct of the group A, B is their free product $A * B$. This construction is discussed systematically later on (see *******), but for now a sketchy description should be enough. The elements of $A*B$ are strings whose elements are taken from either A or B, with obvious identifications such as $xyz \cdots st \cdots uvw = xyz \cdots (st) \cdots uvw$ if $s, t \in A$,

or $s, t \in B$. Also, $\cdots xyzss^{-1}uvw \cdots = \cdots xyzuvw \cdots$ and so on. No identification is made between elements of A and B, even if $A = B$. (Thus, for example $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is an infinite group; some care should be taken with the notation here. It is better to write this group is $A * B$, with $A = \{e, a\}, B = \{e, b\}$. The elements $a, ab, aba, abab, ababa, \ldots$ are all distinct.)

Exercise 2.4.4. Do there exist products and coproducts in **SES**? in **Ring**? in **GSet**?

2.5. **Direct and inverse limits.** The notions of direct and inverse limits can be discussed in much greater generality than what follows. We have chosen to restrict ourselves from the outset to the cases most prevalent in applications.

Let I be a partially-ordered set. We may view I as a category $\underline{I}$: the objects of $\underline{I}$ are the elements of I and if $x \leq y$ then $\mathrm{Mor}(x, y)$ has a single element, call it i_{xy} and else is empty.

Let $\mathbf{C}$ be a category and I a partially ordered set, the index set. A direct system indexed by I is a functor $\underline{I} \to \mathbf{C}$. Namely, for every $i \in I$ an object C_i of $\mathbf{C}$ is given, and for every pair $i \leq j \in I$ a morphism $f_{ij} : C_i \to C_j$ is specified such that $f_{ii} = 1$ and if $i \leq j \leq k$ then $f_{jk} \circ f_{ij} = f_{ik}$.

An inverse system indexed by I is a contravariant functor $\underline{I} \to \mathbf{C}$. Namely, for every $i \in I$ an object C_i of $\mathbf{C}$ is given, and for every pair $i \geq j \in I$ a morphism $f_{ij} : C_i \to C_j$ is specified such that $f_{ii} = 1$ and if $i \geq j \geq k$ then $f_{jk} \circ f_{ij} = f_{ik}$.

The injective (or direct) limit of this data is an object C of $\mathbf{C}$ and morphisms $e_i : C_i \to C$ such that $e_j \circ f_{ij} = e_j$ whenever $i \leq j \in I$ and such that given an object D and morphisms $e_i' : C_i \to D$ such that $e_j' \circ f_{ij} = e_i'$, there is a unique morphism $q : C \to D$ such that $q \circ e_i = e_i'$ for all $i \in I$. It's denoted $\varinjlim C_i$.

$$
\begin{tikzcd}
 & & C_k \arrow[ddl, "e_k"] \arrow[dddl] \\
C_i \arrow[urr, "f_{ik}"] \arrow[r, "f_{ij}"] \arrow[dr, "e_i"] \arrow[ddr] & C_j \arrow[ur, "f_{jk}"] \arrow[d, "e_j"] \arrow[dd, bend right] & \\
 & \varinjlim C_i \arrow[d, dotted] & \\
 & D &
\end{tikzcd}
$$

The projective (or inverse) limit of this data is an object C of $\mathbf{C}$ and morphisms $p_i : C \to C_i$ such that $f_{ij} \circ p_i = p_j$ whenever $i \geq j \in I$ and such that given an object D and morphisms $p_i' : C \to C_i$ such that $f_{ij} \circ p_i' = p_j'$, there is a unique morphism $q : D \to C$ such that $p_i \circ q = p_i'$ for all $i \in I$. It's denoted $\varprojlim C_i$.

$$
\begin{tikzcd}
 & D \arrow[d, dotted] \arrow[dddl, bend right] \arrow[ddd, bend left] \arrow[ddr, bend left] & \\
 & \varprojlim C_i \arrow[ddl, "e_i"'] \arrow[dd, "e_j"] \arrow[dr, "e_k"] & \\
 & & C_k \\
C_i \arrow[r, "f_{ij}"] \arrow[urr, "f_{ik}"] & C_j \arrow[ur, "f_{jk}"'] &
\end{tikzcd}
$$

Theorem 2.5.1. *Direct and inverse limits exist in the category ${}_R Mod$.*

Proof. Consider $\oplus_{i\in I} M_i$ and let $\lambda_i : M_i \to \oplus_{i\in I} M_i$ be the natural inclusion map in the ith coordinate. For the direct limit we take the module $C = \oplus_{i\in I} M_i/W$, where W is the submodule generated by all elements $\{\lambda_i(a) - \lambda_j(f_{ij}(a)) : a \in M_i, i \leq j \in I\}$. The maps $e_i : M_i \to C$ are just $\lambda_i(\cdot) + W$. Since mod W we have $\lambda_i(a) = \lambda_j(f_{ij}(a))$, it follows that $e_i = e_j \circ f_{ij}$. Given an object D with morphism $e_i' : C_i \to D$ define a map $\oplus_{i\in I} M_i \to D$ so that $\lambda_i(a) = e_i'(a)$ for $a \in M_i$. There is a unique such map. Moreover, $\lambda_j(a) = e_j'(a) = e_i'(f_{ij}(a)) = \lambda_i(f_{ij}(a))$ and so we get a well define map

$$q : C \to D,$$

such that $q(\lambda_i(a) \mod W) = e_i'(a)$, that is $q(e_i(a)) = e_i'(a)$ for $a \in M_i$.

The proof for the inverse limit is very similar: "we dualize the proof" given above. We now let $C \subset \prod_{i\in I} M_i$ to consist of all the vectors (m_i) such that $f_{ij}(a_i) = a_j$ whenever $i \geq j$. It is indeed an R-submodule and there are natural projection maps $p_i : C \to M_i$ satisfying $f_{ij} p_i = p_j$. The verification of the universal property is entire similar to what we did above in the case of direct limits. $\square$

The fun in this subjects is definitely in the examples. To begin with products and coproducts are projective and injective limits, respectively, associated to the diagram with no arrows

$$A \qquad B.$$

2.5.1. *Pull-back and push-out.* Let A, B, C be R-modules. Consider the diagram

$$
\begin{tikzcd}
 & B \arrow[d, "\beta"'] \\
A \arrow[r, "\alpha"] & C
\end{tikzcd}
$$

as a projective system. The projective limit P is called in the case the pullback and we have the following diagram:

$$\begin{tikzcd}
P' \arrow[dr, dashed, "q"] \arrow[drr, bend left, "p'_B"] \arrow[ddr, bend right, "p'_A"'] & & \\
& P \arrow[r, "p_B"] \arrow[d, "p_A"'] & B \arrow[d, "\beta"] \\
& A \arrow[r, "\alpha"] & C.
\end{tikzcd}$$

The general theorem we proved gives

$$P = \{(a, b) \in A \times B : \alpha(a) = \beta(b)\},$$

with p_A, p_B being the projections onto the first and second coordinates, respectively. We remark that the same diagram in a geometric, or topological context, is called a fiber product and P is usually denoted $A \times_C B$.

The pushout P is the injective limit of the diagram:

$$\begin{tikzcd}
C \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \\
A & .
\end{tikzcd}$$

We have

$$\begin{tikzcd}
C \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \arrow[d, "e_B"] \arrow[ddr, bend left, "e'_B"] & \\
A \arrow[r, "e_A"'] \arrow[drr, bend right, "e'_A"'] & P \arrow[dr, dashed, "q"] & \\
& & P'.
\end{tikzcd}$$

The pushout is given as

$$P = A \oplus B/M, M = \{(\alpha(c), -\beta(c)) : c \in C\}$$

and the maps e_A, e_B are given by $a \mapsto (a, 0) \mod M, b \mapsto (0, b) \mod M$, respectively.

2.5.2. *Some important examples.* Let R be a ring and $I \lhd R$ an ideal. We have then a natural projective system:

$$\cdots \to R/I^3 \to R/I^2 \to R/I \to 0.$$

The projective limit is called the I-adic completion of R and denoted $R^{\hat{I}}$. It has the following description:

$$\{(\cdots, r_3, r_2, r_1) : r_i \in R/I^i, r_{i+1} = r_i \pmod{I^i}\}.$$

There is a natural map $R \to R^{\hat{I}}$ whose kernel is $\cap I^n$, which may or may not be the zero ideal.

Here are two very interesting examples:

(1) $R = \mathbb{Z}, I = p\mathbb{Z}$. The projective limit is denoted $\mathbb{Z}_p$ and called the ring of p-adic integers. It has the description

$$\{(\cdots, r_3, r_2, r_1) : r_i \in \mathbb{Z}/p^i\mathbb{Z}, r_{i+1} \equiv r_i \pmod{p^i}\}.$$

An interesting point is that although each $\mathbb{Z}/p^i\mathbb{Z}$ is torsion, $\mathbb{Z}_p$ is torsion free.

Exercise 2.5.2. Prove that indeed $\mathbb{Z}_p$ is a ring with a unit and that $\mathbb{Z} \subset \mathbb{Z}_p$. Prove that every integer prime to p is invertible in $\mathbb{Z}_p$. Define a function

$$v : \mathbb{Z}_p \to \mathbb{N}, \qquad v((\ldots, r_3, r_2, r_1)) = \max\{i : r_i = 0\}.$$

Prove that v is a discrete valuation, namely, it satisfies $v(xy) = v(x) + v(y)$ and $v(x + y) \geq \min\{v(x), v(y)\}$ with equality if $v(x) \neq v(y)$. If $z \in \mathbb{Z}$ is viewed as an element of $\mathbb{Z}_p$, what is $v(z)$?

Show that an element of $\mathbb{Z}_p$ is invertible if and only if its valuation is 0. Show that every ideal J of $\mathbb{Z}_p$ is principal, generated by any element of minimal valuation. Prove that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$ in the sense that for any element x of $\mathbb{Z}_p$ there is a series of elements $x_n \in \mathbb{Z}$ such that $v(x - x_n) \to {}_n\infty$; if we define a norm by $\|x\| = p^{v(x)}$ then we have $\|x - x_n\| \to 0$. Prove that relative to the metric defined by the norm $\mathbb{Z}_p$ is compact.

(2) Let k be a field and $R = k[x], I = (x)$. Prove that there is a natural isomorphism

$$\varprojlim k[x]/(x^n) \cong k[[x]],$$

where $k[[x]]$ is the ring of Taylor series with k coefficients.

In algebraic geometry the process of completion allows one to pass to infinitesimal study of varieties with no need to have limits in the sense of classical analysis.