ALGEBRAIC GROUPS: EXERCISES

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Please submit the exercises marked with $\star$. The other exercises are equally important.

(1) Prove that over the complex numbers the group $U(p,q)$ is isomorphic to the unitary group $U(n)$.

(2) Prove that $G(V,q)$ and $G(V,q)^+$ are algebraic groups over k.

(3) $\star$ Assume that k is a field and $\text{char}(k) \neq 2$. In each of the following cases give a concrete model for $\text{Cliff}(V,q)^+$ (as algebras), for $G(V,q), G(V,q)^+$, $\text{Spin}(V,q)$, $S\mathcal{O}_q$ and the homomorphism $\text{Spin}(V,q) \to S\mathcal{O}_q$.

   (a) $V = k$ and $q(x) = tx^2$ for some fixed $t \in k$.

   (b) V is two dimensional over k with the quadratic form $ax^2 + by^2$.

   (c) $V = \mathbb{R}^3$ and $q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$.

(4) Let X be a quasi-projective variety. Prove that every constructible set contains on open dense set of its closure.

(5) Find a morphism $\mathbb{A}^2 \to \mathbb{A}^2$ whose image is the set $(\mathbb{A}^2 - \{x = 0\}) \cup \{(0,0)\}$.

(6) If H and K are closed subgroups of G, one of which is connected then (H,K) - the subgroup of G generated by all the commutators $xyx^{-1}y^{-1}, x \in H, y \in K$ - is closed and connected.

(7) $\star$ Prove that the symplectic group is connected. You may want to use transvections for that.

(8) $\star$ Let $GL_2(\mathbb{C})$ act on $M_2(\mathbb{C})$ by conjugation. Determine the orbits of this action using the Jordan form. Determine the closure of an orbit and, in particular, find all the closed orbits.

(9) Show that an action of $\mathbb{G}_a$ on the affine variety $\mathbb{A}^1 - \{0\}$ must be trivial.

(10) Prove that if $a \in \text{End}(V), b \in \text{End}(W)$, where V, W are finite dimensional k-vector spaces, are semisimple (nilpotent, unipotent) then so is $a \oplus b \in \text{End}(V \oplus W)$ and $a \otimes b \in \text{End}(V \otimes W)$.

Date: Winter 2010.

(11) Let G be a subgroup of GL_n that acts irreducibly on k^n. Prove that the only normal unipotent subgroup of G is the trivial one.

(12) Let U_2 be the standard unipotent group in GL_2. Find the orbits of U_2 in its action on k^2. Observe that they are indeed closed.

(13) Give an example of an action of U_2 on a projective algebraic variety such that not all orbits are closed.

(14) Do exercises (2)-(3) on p. 48 of Springer's book.

(15) In the setting of the previous exercise. Let $\mathbb{G}_m$ acts on $\mathbb{A}^2$ by $t.(v_1, v_2) = (t^{a_1}v_1, t^{a_2}v_2)$, where a_1, a_2 are some fixed integers. What is the decomposition of $\mathbb{A}^2$?

(16) $\star$ Let k be an algebraically closed field of characteristic p. Show that there is an anti-equivalence of categories between the category of finitely generated abelian groups with no p-torsion and diagonalizable k-groups. This antiequivalence assoicate $X^*(G)$ to a diagonalizable group G. Show, further, that $G_1 \to G_2$ is injective (resp. surjective) iff $X^*(G_2) \to X^*(G_1)$ is surjective (resp. injective).

(17) Show that every one parameter subgroup of GL_n is conjugate to one of the form $x \mapsto \mathrm{diag}(x^{a_1}, \dots, x^{a_n})$ where $a_1 \geq a_2 \geq \cdots \geq a_n$ are integers. Determine $P(\lambda)$ and the centralizer of λ.

(18) $\star$ Consider the closed subgroup H of GL_2 consisting of matrices of the from $\left(\begin{smallmatrix} t_1 & t_2 \\ 0 & 1 \end{smallmatrix}\right)$. Determine explicitly the left invariant derivations of H. What are the derivations corresponding to the point derivations $f \mapsto \frac{\partial f}{\partial t_1}(e), f \mapsto \frac{\partial f}{\partial t_2}(e)$?

(19) A maximal torus T of a linear algebraic group G is a torus T contained in G that is not strictly contained in another torus of G. Working over an algebraically closed field, find a maximal torus for the groups GL_n, SL_n, $\mathrm{SO}_{2\mathrm{n}}, \mathrm{SO}_{2\mathrm{n}+1}$, Sp_{2n} and $\mathrm{Spin}_{2\mathrm{n}}$. You may use the fact that such a torus will have rank n in all cases, except for SL_n where the rank is $n-1$. For the group GL_n prove maximality without using that.

(20) Find the Lie algebra of Sp_{2n}.

(21) Prove that the Lie algebra of $\mathrm{Spin}(V, q)$ is isomorphic to the Lie algebra of SO_q.

(22) Let G_1, G_2 be linear algebraic groups. Prove that $\mathscr{L}(G_1 \times G_2) \cong \mathscr{L}(G_1) \times \mathscr{L}(G_2)$.

(23) Let k be algebraically closed. Let G be a torus over k. Prove that there is a canonical isomorphism $\mathfrak{g} \cong X_*(G) \otimes_{\mathbb{Z}} k$ (in particular, show that this isomorphism is compatible with maps between tori).

(24) Calculate the adjoint representation $\mathrm{Ad} : \mathrm{SL}_2 \to \mathrm{GL}_3$ and $\mathfrak{ad} : \mathfrak{sl}_2 \to \mathfrak{gl}_3$ with respect to the basis $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

(25) ° Let G be a linear algebraic group and B a Borel subgroup of G. Let $\sigma : G \to G$ be an automorphism such that $\sigma(b) = b, \forall b \in B$. Prove that σ is the identity.
(26) Find a Borel subgroup of Symp_{2n}.
(27) ° Find a proper parabolic subgroup of SO_n.
(28) ° Find a Borel subgroup of SO_n.
(29) ° Let $\phi : G \to H$ be a surjective homomorphism of linear algebraic groups. Is the preimage of a parabolic? what about Borel? What happens if we drop the assumption of surjective?
(30) ° Let G be a connected algebraic group such that every element of G is semisimple. Prove that G is a torus.
(31) ° Prove that the commutator subgroup of $\mathbb{T}_n$ is $\mathbb{U}_n$. Calculate the ascending central series of $\mathbb{U}_n$.
(32) ° Let $m : G \times G \to G$ be multiplication and $i : G \to G$ be inversion. Prove that $dm_{(e,e)}(X, Y) = X + Y$ and $di_e(X) = -X$.
(33) ° Classify the centralizers of semi-simple elements of GL_n in terms of their characteristic polynomial (more specifically, the multiplicities of roots).
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