ALGEBRAIC GROUPS: PART V

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14. General structure theorems for connected algebraic groups

Let G be a connected linear algebraic group. By a **maximal torus** of G we mean a torus of G not properly contained in any other torus.

Theorem 14.0.1. *Let G be a connected linear algebraic group. Any two maximal tori in G are conjugate.*

Proof. Every maximal torus, being connected and solvable, is contained in a Borel subgroup. We proved that all Borel subgroups are conjugate and all the maximal tori of a Borel subgroup are conjugate (in that Borel subgroup). ☐

Definition 14.0.2. Let G be a connected linear algebraic group and let T be a maximal torus in G. Then $\dim(T)$ is called the **rank** of G. It is independent of the choice of T.

Example 14.0.3. $\mathrm{rk}(\mathrm{GL}_n) = n$, $\mathrm{rk}(\mathrm{SL}_n) = n - 1, \mathrm{rk}(\mathrm{Sp}_{2n} = n$, $\mathrm{rk}(\mathbb{T}_n) = n$.

Proposition 14.0.4 (Rigidity of Tori)**.** *Let G and H be two diagonalizable groups and let V be a connected affine variety. Assume given a morphism*

$$\phi : V \times G \to H,$$

such that for any $v \in V$ the map $x \mapsto \phi(v, x)$ is a homomorphism of algebraic groups $G \to H$. Then $\phi(v, x)$ is independent of v. (Colloquially, a family of homomorphisms $G \to H$ indexed by a connected variety V is constant.)

The proposition states that one cannot continuously deform homomorphisms of diagonalizable groups. This makes sense heuristically as morphism of diagonalizable groups are determined by their effect on characters groups, which are discrete objects.

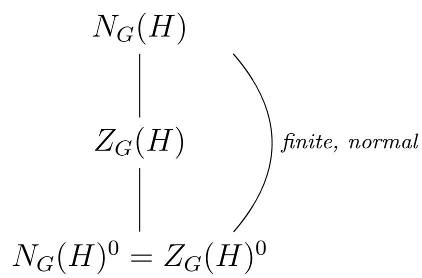
Proof. Let $\psi \in X^*(H)$ be a character of H. It is in particular a regular function on H and so

$$\phi^*\psi(v, x) = \sum_{\chi \in X^*(G)} f_{\chi,\psi}(v)\chi(x),$$

with $f_{\chi,\psi} \in k[V]$. Here we have used that $k[V \times G] = k[V] \otimes_k k[G]$ and that $X^*(G)$ is a basis for $k[G]$ over k.

By our assumption, for a fixed v the sum $\sum_{\chi \in X^*(G)} f_{\chi,\psi}(v)\chi(x)$ is a character of G, because $x \mapsto \phi(v, x)$ is a homomorphism of algebraic groups. By Dedekind's independence of characters, we must have $f_{\chi,\psi}(v) = 1$ for exactly one χ and zero for all the other χ. For a fixed χ, the conditions $f_{\chi,\psi}(v) = 1$, or $f_{\chi,\psi}(v) = 0$ are closed conditions and as v varies over an irreducible component of V they exhibit the component as a disjoint union of two closed sets, thus one of them must be empty. For every irreducible component choose v and χ such that $f_{\chi,\psi}(v) = 1$. Then $f_{\chi,\psi}(v) = 1$ for any v in that irreducible component and $f_{\chi',\psi}(v) = 0$ for any v in that component and any $\chi' \neq \chi$. The connectedness of V now implies that for that χ, $f_{\chi,\psi} \equiv 1$ and for any $\chi' \neq \chi$, $f_{\chi',\psi} \equiv 0$. □

Proposition 14.0.5. *Let G be a linear algebraic group and H a diagonalizable subgroup of G. Then (i) $N_G(H)^0 = Z_G(H)^0$, (ii) $Z_G(H)$ and $Z_G(H)^0$ are normal in $N_G(H)$, and (iii) the groups $N_G(H)/Z_G(H)$, $N_G(H)/Z_G(H)^0$, are finite.*



Proof. Let $z \in Z_G(H), n \in N_G(H)$ and $h \in H$. Then $(nzn^{-1})h = nz(n^{-1}hn)n^{-1} = n(n^{-1}hn)zn^{-1} = h(nzn^{-1})$ which proves that $Z_G(H)$ is a normal subgroup of $N_G(H)$. Since $Z_G(H)^0$ is a characteristic subgroup of $Z_G(H)$ it is normal in $N_G(H)$ too (and this will also follow from $N_G(H)^0 = Z_G(H)^0$).

Apply the previous proposition with

$$N_G(H)^0 \times H \to H, \qquad (n, h) \mapsto nhn^{-1}.$$

This map is therefore independent of n. For $n = 1$ it is the identity. Thus, $N_G(H)^0 \subset Z_G(H)$ and so $N_G(H)^0 \subset Z_G(H)^0$; the other inclusion being obvious, we get $N_G(H)^0 = Z_G(H)^0$.

The group $N_G(H)/N_G(H)^0$ is finite and $N_G(H)/Z_G(H)$ is a quotient of it. □

The following Lemma is of independent interest. It is also useful in practice in calculating centralizers of tori, as it reduces the calculation to calculating the centralizer of a single element that the proof explains how to choose.

Lemma 14.0.6. *Let G be a linear algebraic group and S a torus of G. There exists $s \in S$ such that $Z_G(s) = Z_G(S)$.*

Proof. Embed G in GL_n so that S are diagonal matrices. We may then assume, without loss of generality, that $G = \mathrm{GL}_n$. S being diagonal, its diagonal entries are characters $s \in S \mapsto s_{ii}$. Let $\chi_1, \dots, \chi_m$ be the distinct characters obtained this way. Let $s_0 \in S$ be an element such that $\chi_i(s_0) \neq \chi_j(s_0)$ for $i \neq j$. Such s_0 exists, because each of the finitely many characters χ_i/χ_j is equal to 1 on a codimension 1 subtorus of S.

The inclusion $Z_G(s) \supset Z_G(S)$ is obvious. We may arrange the coordinates so that

$$S = \{\mathrm{diag}(\underbrace{\chi_1(s), \dots \chi_1(s)}_{a_1}, \underbrace{\chi_2(s), \dots, \chi_2(s)}_{a_2}, \cdots, \underbrace{\chi_m(s), \dots \chi_m(s)}_{a_m}) : s \in S)\}.$$

Then one can easily check that $Z_G(s) \cong \mathrm{GL}_{a_1} \times \mathrm{GL}_{a_2} \times \cdots \times \mathrm{GL}_{a_m}$ and that this group centralizes S as well. (*Remember that this is the centralizer in* GL_n. *If one wants to apply that to the original G, it is the intersection with the image G under an embedding diagonalizing S in a very specific manner.*) □

Example 14.0.7. Consider the torus $\{\mathrm{diag}(t_1, \dots, t_m, 1, \dots, 1)\}$ in GL_n. Its centralizer is therefore $\{\mathrm{diag}(t_1, \dots, t_m)\} \times \mathrm{GL}_{n-m}$.

Let G be a connected linear algebraic group. A **Cartan subgroup** of G is the identity component of the centralizer of a maximal torus of G. (We shall see later that in fact the centralizer of a maximal torus is already connected.)

Proposition 14.0.8. *Let G be a connected algebraic group. Let T be a maximal torus and $C = Z_G(T)^0$ the corresponding Cartan subgroup. Then:*

(1) *C is nilpotent and T is its unique maximal torus. In particular, C is contained in some Borel subgroup.*
(2) *There exists elements $t \in T$ lying in only finitely many conjugates of C.*

Proof. We will need a lemma.

Lemma 14.0.9. *Let G be an algebraic group and B a Borel subgroup of G. If B is nilpotent then $B = G^0$.*

Proof of Lemma. The proof is by induction on $\dim(G)$. The case $\dim(G) = 0$ is obvious. In general, since B is nilpotent, B has a non-trivial closed connected central subgroup J, for example that generated by commutators of maximal length that are not yet trivial. Since $C(B) \subseteq C(G)$, J is normal in G. We may then pass to G/J and B/J (the image of Borel is Borel, as we have proven) and conclude by induction. □

Now, C contains T as a central subgroup; choose a Borel subgroup B of C containing T. Then T is a central subgroup of B. In this case the isomorphism of varieties $T \times B_u \to B$ is also an isomorphism of algebraic groups and so, since B_u is nilpotent, B is nilpotent. By the lemma $B = C$. In particular, C is nilpotent and T is its central torus, $C = T \times C_u$.

To prove (2) we choose an element $t \in T$ such that $Z_G(t) = Z_G(T)$. If $t \in gCg^{-1}$ then $g^{-1}tg \in C = Z_G(T)^0$ and so $T \subset Z_G(g^{-1}tg) = Z_G(g^{-1}Tg)$ and must be the maximal torus of $Z_G(g^{-1}Tg)$. But, $Z_G(g^{-1}Tg)$ also contains $g^{-1}Tg$ as a maximal torus. Thus, $T = g^{-1}Tg$ and hence $g \in N_G(T)$. As $N_G(T)/Z_G(T)$ is finite, there are only finitely many conjugates gCg^{-1} containing t (note that gCg^{-1} depends only on the coset $gZ_G(T)$ [1]). □

The next theorem is a very important theorem. It explains the special role played by tori and Borel subgroups in the study of linear algebraic groups.

Theorem 14.0.10. *Let G be a connected linear algebraic group.*

(1) *Every element of G lies in some Borel subgroup.*
(2) *Every semisimple element of G lies in a maximal torus.*
(3) *The union of the Cartan subgroups of G contains a dense open subset of G. (So "almost" every element of G lies in a Cartan subgroup.)*

Before the proof we need a lemma.

Lemma 14.0.11. *Let H be a closed subgroup of a connected linear group G. Let $X = \cup_{g\in G} gHg^{-1}$ and $\bar{X}$ the Zariski closure of X.*

(1) *X contains a non-empty open subset of $\bar{X}$. If H is parabolic then $X = \bar{X}$.*
(2) *Assume that H has a finite index in its normalizer and that there exist elements of H lying in only finitely many conjugates of H. Then $\bar{X} = G$.*

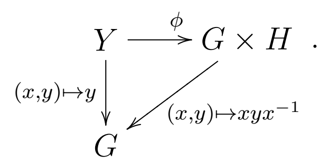
[1] We are not claiming though that distinct cosets give distinct conjugates. This would be the case of $Z_G(T) = N_G(T)$, but this is almost never the case.

Proof of lemma. We may assume H is connected. We can view X as the image of $G \times H$ under the morphism $G \times H \to G, (g,h) \mapsto ghg^{-1}$. It follows that X is constructible, hence contains a non-empty open subset of $\bar{X}$.

It is also useful to view X as the image of a different morphism. Consider the isomorphism $\phi : G \times G \to G \times G$ given by

$$\phi(x, y) = (x, x^{-1}yx).$$

Let $Y = \{(x, y) : x^{-1}yx \in H\}$. Then $Y = \phi^{-1}(G \times H)$. We have a commutative diagram



Y is a closed subset of $G \times G$ and it is H-closed; that is, if $h \in H, (x, y) \in Y$ then $(xh, y) \in Y$. Indeed, $\phi(xh, y) = h^{-1}(x^{-1}yx)h \in H$. Therefore, if H is parabolic, then the image of Y under the projection $p_2 : G \times G \to G$, namely X, is closed in G.

We may now let $\bar{Y}$ be the image of Y in $G/H \times G$. Since Y is closed and equal to the pre-image of $\bar{Y}$, we can conclude from $G \to G/H$ being an open map that $\bar{Y}$ is a closed set of $G/H \times G$, hence a variety. $\bar{Y}$ is irreducible, being the image of $Y \cong G \times H$. We have

$$\dim(\bar{Y}) = \dim Y - \dim H = \dim(G).$$

We have an induced map

$$\bar{p}_2 : \bar{Y} \to G, \qquad (xH, y) \mapsto y.$$

Let y be an element of H that belongs to finitely many cosets of H, say $t_1Ht_1^{-1}, \dots, t_mHt_m^{-1}$. The preimages of y in Y are the elements (x, y) such that $x^{-1}yx \in H$, or, $y \in xHx^{-1}$. Such x is therefore of the form $x = t_in$ for some $n \in N_G(H)$. It follows that the preimages of y in $\bar{Y}$ are the elements $(t_i\alpha_jH, y)$ where the α_j are the finite number of cosets representatives for $N_G(H)/H$. In particular, the fiber of $\bar{Y} \to X$ over y is finite, hence zero-dimensional. Thus, the fibre is zero-dimensional over almost any point of X. It follows that

$$\dim(X) = \dim(\bar{Y}) = \dim(G).$$

Thus, $\bar{X} = G$. □

Proof of the theorem. Let T be a maximal torus and $C = Z_G(T)^0$ the Cartan subgroup. Suppose that $x \in N_G(C)$ then x must also conjugate the only maximal torus of C

(Proposition 14.0.8) to itself. Thus, $x \in N_G(T)$. On the other hand, $N_G(T)$ normalizes $Z_G(T)^0 = N_G(T)^0$,as we have shown above (Proposition 14.0.5). Thus,

$$\boxed{N_G(C) = N_G(T)}$$

It follows that $C = Z_G(T)^0$ has finite index in $N_G(C)$ and so if we apply the Lemma with $H = C$, the conditions of part (ii) hold, because we had also proven that there is an element of T (hence of C) that lies in only finitely many conjugates of C. Thus, we conclude that

$$G = \text{Zariski closure}(\cup_{x \in G} xCx^{-1}),$$

and (3) follows.

Now, we know C is connected and nilpotent, hence solvable. Thus C is contained in some Borel subgroup B. Since a Borel is parabolic, the Lemma gives that $\cup_{x \in G} xBx^{-1}$ is a closed set. It contains the dense set $\cup_{x \in G} xCx^{-1}$. Thus,

$$G = \cup_{x \in G} xBx^{-1}.$$

Finally, let s be a semisimple element of G. Then s lies in some Borel B and we can apply Theorem 13.3.5 to deduce that x lies in a maximal torus of B (so of G). □

Corollary 14.0.12. *Let B be a Borel subgroup of a connected linear algebraic group G. Then*

$$C(B) = C(G).$$

Proof. We had already proven that $C(B) \subseteq C(G)$. Let $g \in C(G)$; g belongs to some Borel subgroup. Since all Borel subgroups are conjugate and g is fixed by conjugation, g belongs to all Borel subgroups. Thus, $g \in C(B)$. □

Theorem 14.0.13. *Let S be a subtorus of a connected linear algebraic group G.*

(1) *The centralizer $Z_G(S)$ is connected.*
(2) *If B is a Borel subgroup containing S then $Z_G(S) \cap B$ is a Borel subgroup of $Z_G(S)$. Every Borel subgroup of $Z_G(S)$ is obtained this way.*

Proof. We will only prove here the first part. The proof of the second part is completely within our means now, but in the interest of time we don't give it. See Springer's theorem 6.4.7 for the proof.

Let $g \in Z_G(S)$ and let B be a Borel subgroup containing g. Let

$$X = \{xB \in G/B : x^{-1}gx \in B\}.$$

Note that $X = (\bar{p}_2)^{-1}(g)$ in the notation of Lemma 14.0.11 (used for $H = B$), and so X is a closed subset of G/B. Thus, X is a complete variety. The torus S acts on X by $(s, xB) \mapsto sxB$ (since g centralizes S). Using Borel's fixed point theorem, there is an element xB of X that is fixed by the action of S: $sxB = xB, \forall s \in S$. That is, $x^{-1}Sx \subset B$.

Since B is connected solvable, we can apply Corollary 13.3.8 to $H = x^{-1}Sx$ and conclude that $x^{-1}gx$, which is in the centralizer of $x^{-1}Sx$ and also in B (by definition of X) belongs to the identity component of $Z_B(x^{-1}Sx)$. Thus, g belongs to the identity component of $Z_{xBx^{-1}}(S)$ and hence to the identity component of $Z_G(S)$. We proved that $Z_G(S)$ is connected. □

Corollary 14.0.14. *Let S be a torus of a connected linear algebraic group G. Then $Z_G(S) = Z_G(S)^0 = N_G(S)^0$ and is of finite index in $N_G(S)$. In particular, this holds for a maximal torus T. Thus, a Cartan subgroup is the centralizer of a maximal torus (no need to add anymore that it is the connected component of that centralizer). The finite group*

$$N_G(T)/N_G(T)^0 = N_G(T)/Z_G(T) =: W(G, T)$$

is called the **Weyl group** *of G relative to T. Any two Weyl groups of G are conjugate (because maximal tori are), hence isomorphic, but not canonically.*[2]

Corollary 14.0.15. *Let B be a Borel subgroup containing a maximal torus T then B contains its Cartan subgroup C.*

Proof. The theorem tells us that $B \cap C$ is a Borel subgroup. However, if T is a maximal torus, its centralizer C is a nilpotent subgroup (Proposition 14.0.8), hence solvable. Thus, every Borel of C is equal to C. That is, $B \supseteq C$. □

We come now to a very important theorem in the structure of algebraic groups. Unfortunately, its proof employs techniques appearing in parts of Springer's book that we didn't cover. Thus, we omit the proof. But see the remark following the theorem for a weaker statement.

[2] One could say canonically up to inner automorphism. However such groups often contain a large symmetric group and so the distinction between "automorphism" and "inner automorphism" is not so important.

Theorem 14.0.16. *Let G be a connected linear algebraic group and B a Borel subgroup of G. Then,*

$$N_G(B) = B.$$

Remark 14.0.17. We can at least show that $B = N_G(B)^0$ and thus see why it is of finite index in $N_G(B)$.

The group $N_G(B)^0$ is connected and B is its Borel subgroup; in fact, its only Borel subgroup, since all Borel subgroups are conjugate. But, every element of $N_G(B)^0$ belongs to some Borel subgroup. Thus, $N_G(B)^0 = B$.

Corollary 14.0.18. *Let G be a connected linear algebraic group. Let P be a parabolic subgroup of G. Then P is connected and $N_G(P) = P$.*

Proof. P contains a Borel subgroup B and, in fact, $B \subset P^0$. Let $x \in N_G(P)$ then xBx^{-1} is another Borel subgroup of the connected linear algebraic group P^0 and so - since all Borels are conjugate in P^0, there is a $y \in P^0$ such that $xBx^{-1} = yBy^{-1}$. Then, $y^{-1}x \in N_G(B) = B$ and so $x \in yB \subset P^0$. That is, $P^0 \supset N_G(P) \supset P \supset P^0$ and everything follows. □

Corollary 14.0.19. *Let P be a parabolic subgroup of a connected linear algebraic group G. Then,*

$$Z_G(P) = C(P) = C(G).$$

Proof. Let P be a parabolic subgroup. Then, since $Z_G(P) \subset N_G(P) = P$, we find that $Z_G(P) = C(P)$ is the centre of P. Let B be a Borel subgroup contained in P. Then, as we proved before, $C(B) = C(G)$ and so $C(B) \subseteq C(P)$. On the other hand, certainly $C(P) = Z_G(P) \subset Z_G(B) = C(B)$. Thus, $C(P) = C(B) = C(G)$. □

15. Summary of some results so far

Let G be a connected algebraic group. We have 3 notions that are prominent so far: (i) a maximal torus; (ii) a Borel subgroup; (iii) a parabolic subgroup.

We also have three "operations": (i) take the centralizer; (ii) take the normalizer; (iii) conjugate.

It is interesting to examine the knowledge we have so far in light of these concepts and operations. We know the following.

(1) Every element belongs to some Borel subgroup.
(2) Every semisimple element belongs to some maximal torus.

As to conjugation:

(1) A conjugate of a maximal torus is a maximal torus and any two maximal tori are conjugate. In particular, they all have the same dimension. Every maximal torus is contained in some Borel B, hence the dimension of maximal tori is $\dim(B/B_u)$.
(2) A conjugate of a Borel is a Borel and all Borel are conjugate.
(3) A conjugate of a parabolic is parabolic.

As to inclusions:

(1) Every parabolic contains a Borel. Every Borel is parabolic.
(2) Every Borel contains a maximal torus, every maximal torus is contained in some Borel.
(3) The centralizer of a maximal torus T - a Cartan subgroup $C = C(T)$ - is connected, nilpotent, and is equal to $N_G(T)^0$ and is contained in every Borel containing T. We have $N_G(C) = N_G(T)$.
(4) The normalizer of a parabolic is equal to the parabolic. The centralizer of a parabolic is equal to its centre and is equal to the centre of the ambient group G.

It is now interesting to look at collections.

(1) The collection of Borel subgroups of G is in bijection with the projective variety G/B, where B is some Borel. More generally, the collection of Borel subgroups contained in a given parabolic subgroup P is in bijection with the projective variety P/B, where B is some Borel of P.

(2) The collection of maximal tori of G is in bijection with $G/N_G(T)$, which has a finite cover by $G/Z_G(T)$ (often $T = Z_G(T)$). The collection of maximal tori contained in a fixed Borel subgroup is in bijection with $B/N_B(T) = B/Z_B(T)$.
(3) The collection of Borel subgroups containing a given maximal torus T is in bijection with $N_G(T)/Z_G(T) = W(G,T)$ - the Weyl group (which is a finite group).
(4) Let B be a Borel subgroup. Out of every conjugacy class of parabolic subgroups there's exactly one element containing B. In particular, the parabolic subgroups containing B are in bijection with conjugacy classes of parabolic subgroups. These can be classified by means of roots.

We have more or less proven all these assertions. The assertions we didn't prove can be deduced quite easily from those we have proved. For example, one needs that if P, Q are conjugate parabolic containing a Borel B then $P = Q$. Indeed, if $P = xQx^{-1}$ then the Borels B and xBx^{-1} are contained in P and so conjugate in P. Thus, for some $y \in P$, $B = yxBx^{-1}y^{-1}$. Then $yx \in N_P(B) = B$ and so $x \in y^{-1}B \subseteq P$ and so $P = Q$.

All this, while beautiful, doesn't explain completely why the notions we were occupied with are so central. It is the role the play in the classification of algebraic groups and their representations that makes them central to the whole theory.

15.1. **Some definitions.** Let A, B be closed normal subgroups of a connected algebraic group G. Then AB is also a closed normal subgroup, connected if A and B are, solvable if A and B are. Thus, there is a maximal closed connected solvable and normal subgroup of G. This group is called the **radical** of G and will be denoted here $R(G)$. A group is called semi-simple if $R(G) = \{1\}$.

Similarly, if A and B are normal and unipotent then so is AB. Thus, there is a maximal closed connected unipotent normal subgroup of G. This group is called the **unipotent radical** of G and is denoted here $R_u(G)$.

How to calculate these? Note that the radical being connected and solvable is contained in a Borel subgroup. Thus, being normal, it is contained in all Borel subgroups and being connected, $R(G) \subseteq (\cap_{B \text{ Borel}} B)^0$. On the other $(\cap_{B \text{ Borel}} B)^0$ is a closed connected normal solvable subgroup of G, hence contained in the radical. Thus:

$$R(G) = (\cap_{B \text{ Borel}} B)^0.$$

In the same way,

$$R_u(G) = (\cap_{B \text{ Borel}} B_u)^0.$$

Since for every Borel B, $R(G)_u \subseteq B_u$, and being connected (as $R(G)$ is connected solvable) $R(G)_u \subseteq R_u(G)$. On the other hand, $R_u(G)$ is contained in $R(G)$ and consists of unipotent elements, hence $R_u(G) \subseteq R(G)_u$. Thus,

$$\boxed{R_u(G) = R(G)_u}.$$

A connected linear algebraic group G is called **semisimple** if $R(G) = \{1\}$ and **reductive** if $R_u(G) = \{1\}$.

Example 15.1.1. Suppose that the centre of the group is positive dimensional, equivalently, $C(G)^0$ is not trivial. It is easy to see that $C(G)^0 \subseteq R(G)$. Then G cannot be semisimple. For example, we see that GL_n is not semisimple.

On the other hand, GL_n is reductive. We know that $\mathbb{G}_m = C(\mathrm{GL}_n) \subset R(\mathrm{GL}_n)$. The two Borel subgroups $\mathbb{T}_n$ and ${}^t\mathbb{T}_n$ show the radical is contained in $\mathbb{D}_n = \mathbb{T}_n \cap {}^t\mathbb{T}_n$. On the other hand, any diagonal matrix having two non-equal entries is conjugate to a non-diagonal matrix. For example, $\mathrm{diag}(2, 1, a, b, c, \dots)$ is conjugate to $\mathrm{diag}(\left(\begin{smallmatrix} 2 & 1 \\ & 1 \end{smallmatrix}\right), a, b, c, \dots)$. That implies that the largest normal subgroup of GL_n contained in $\mathbb{D}_n$ are the scalar matrices $\mathbb{G}_m$. Thus,

$$R(\mathrm{GL}_n) = k^\times \cdot I_n \cong \mathbb{G}_m, \qquad R_u(\mathrm{GL}_n) = \{1\}.$$

On the other hand, SL_n is semisimple. If G is a closed normal subgroup of H then it is not hard to check that $R(G)$ is a normal subgroup of H (in fact, $R(G)$ is a characteristic subgroup of G and in particular preserved under the automorphisms induced by conjugation by elements of H). Thus, $R(G) \subseteq R(H)$. Thus, $R(\mathrm{SL}_n) \subseteq R(\mathrm{GL}_n) = k^\times \cdot I_n$. But the only elements of determinant one in that group is the finite subgroup of n-th roots of unity. Since $R(\mathrm{SL}_n)$ is connected, it must be trivial.

Example 15.1.2. Let P be a parabolic subgroup in GL_n then P is not reductive. Indeed, if we write P as the matrices

$$M = \begin{pmatrix} A_1 & * & * & \dots & * \\ & A_2 & * & \dots & * \\ & & A_3 & \dots & * \\ & & & \ddots & \\ & & & & A_t \end{pmatrix},$$

(A_i of size a_i) the the subgroup of P where all the A_i are identity matrices is normal, connected, unipotent, hence contained in $R_u(P)$. Indeed, this subgroup is the kernel of the homomorphism $P \to \mathrm{GL}_{a_1} \times \cdots \times \mathrm{GL}_{a_t}$, $M \mapsto \mathrm{diag}(A_1, \ldots, A_t)$.

Proposition 15.1.3. *Let G be a connected linear algebraic group. Then $G/R(G)$ is semisimple. Its rank is called the* **semisimple rank** *of G. Similarly, $G/R_u(G)$ is reductive.*

Proof. Let H be a normal connected closed and solvable subgroup of $G/R(G)$. Then, it preimage in G, say $\tilde{H}$ is a closed connected normal subgroup. It is also solvable, because it sits in an exact sequence $1 \to R(G) \to \tilde{H} \to H \to 1$, where both $R(G)$ and H are solvable. Thus, $\tilde{H} \subset R(G)$ and so $H = \{1\}$.

The proof for $G/R_u(G)$ is the same, where now one needs that if $1 \to R_u(G) \to \tilde{H} \to H \to 1$ is an exact sequence and H is unipotent so is $\tilde{H}$. Indeed, if it had a semisimple element h its projection to H would be both semisimple and unipotent and so trivial. Thus, $h \in R_u(G)$ and again semisimple and unipotent, hence trivial. □

We have the following additional useful results:

- If G is reductive then $R(G)$ is a maximal torus. $G = R(G) \cdot (G, G)$ and $R(G) \cap (G, G)$ is finite. Thus, $R(G) \times (G, G) \to G$ is a surjective homomorphism with a finite kernel (isogeny). Further, (G, G) is semisimple.
- If G is semisimple then $G = (G, G)$ and it has a finite centre.
- If G is reductive and T is a maximal torus of G then $C(T) = T$, that is, Cartan subgroups are tori, and $C(G) \subseteq T$. Further, if S is any torus then $Z_G(S)$ is connected and reductive.
- If G is connected, semi-simple, of rank 1 then $G \cong \mathrm{SL}_2$ or PSL_2.