ALGEBRAIC GROUPS: PART IV

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Contents

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11. Quotients

Very few proofs will be given in this section. The missing proofs can be found in Springer, Humphreys and Borel (and other references that we give below). This is not because there is anything very difficult about them; it is done so that we still have time to discuss other topics.

11.1. **Some general comments.** Let X be a quasi-projective variety and H an algebraic group acting on X. Ideally, we want a quotient X/H. One would expect:

- X/H is a quasi-projective variety.
- There is a morphism $X \to X/H$ that gives a bijection between points of X/H and orbits of H in X.
- Additional properties: one may expect the morphism π to be open, that every morphism $X \to Y$ constant on orbits of H factors through X/H and so on.

Unfortunately, already in very simple cases such a quotient doesn't exists. For example, let $H = \mathbb{G}_m$ act on $X = \mathbb{A}^1$. There are two orbits $\{0\}$ and $\mathbb{A}^1 - \{0\}$. If $\mathbb{A}^1/\mathbb{G}_m$ existed it should thus have two points, corresponding to the two orbits. Since X/H is a variety, each such point is closed and so both orbits $\{0\}$ and $\mathbb{A}^1 - \{0\}$ are closed, which is not the case.

On the other ase, if X is an affine variety and H is a finite group then a quotient exists and moreover, X/H is an affine variety with coordinate ring $k[X/H] = k[X]^H$. So, for example, although it is not obvious, $\mathrm{PSL}_2 = \mathrm{SL}_2/\{\pm 1\}$ is an affine variety. Moreover, the coordinate ring of SL_2 is

$$k[\mathrm{SL}_2] = k[x_{11}, x_{12}, x_{21}, x_{22}, x_{11}x_{22} - x_{12}x_{21} - 1],$$

and so the coordinate ring of PSL_2 is

$$k[x_{ij}x_{kl}, x_{11}x_{22} - x_{12}x_{21} - 1 : i, j, k, l \in \{\pm 1\}].$$

Note that it is presented as a closed subvariety of $\mathbb{A}^{10}$!

The following theorem is very useful. There is a proof in Mumford/Abelian varieties.

Theorem 11.1.1. *Let X be a quasi-projective variety and G a finite group acting on X, then there is a* **quotient** *$X \to X/G$ with all the properties above (including the additional properties).*

11.2. **The quotient of a linear group by a subgroup.** Let G be an affine linear group over an algebraically closed field k. Let H be a closed subgroup of G. The main theorem - a proof can be found in Springer's book - is the following:

Theorem 11.2.1. *There is a* **quotient** G/H *in the following sense: There is a homogenous G-variety, denoted G/H, and a point $a \in G/H$ such that:*

for any pair (Y, b) comprising a homogenous G-variety Y and a point $b \in Y$ such that $\mathrm{Stab}_G(b) \supseteq H$, *exists a unique morphism*

$$G/H \to Y, \qquad a \mapsto b.$$

Sketch of proof.
Step 0. Recall that a variety for Springer is a ringed space $(X, \mathcal{O}_X)$ that is quasi-compact and locally isomorphic to an affine variety. This means the following: X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X (see, e.g., Hartshorne for the concept of "sheaf"). Quasi-compact means that every open cover contains a finite open subcover. (Some people call that "compact"; here we follow the convention that a compact space is quasi-compact and Hausdorff). Every quasi-projective variety is an example of a quasi-compact ringed space. If $(X, \mathcal{O}_X)$ is a ringed space and $U \subset X$ is an open set then $(U, \mathcal{O}_X|_U)$ is a ringed space. The assumption made on $(X, \mathcal{O}_X)$ that it can be covered by open sets U such that each $(U, \mathcal{O}_X|_U)$ is isomorphic as a ringed space to one obtained from a quasi-projective variety (we could have required "from an affine variety" and that wouldn't matter).

This category of variety is larger and more flexible then the category of quasi-projective varieties.

Step 1. There exists a G-homogenous quasi-projective variety X for G and $x \in X$ such that:

- $\mathrm{Stab}_G(x) = H$.
- the morphism $\psi : G \to X, g \mapsto gx$ gives a separable morphism $G^0 \to G^0x$.
- the fibers of ψ are the cosets gH.

[[We recall that a dominant morphism of quasi-projective varieties $\phi : A \to B$ is called separable, if for every $a \in A$ the induced map $T_{A,a} \to T_{B,\phi(a)}$ is surjective. Let k be a field of characteristic $p > 0$. Here is a typical example of a non-separable morphism:

$$\phi : \mathbb{G}_a \to \mathbb{G}_a, \qquad \phi(x) = x^p.$$

This morphism is called the Frobenius morphism. Now, let $\mathbb{G}_a$ act on $\mathbb{A}^1$ via ϕ. That is

$$x \star a = x^p + a.$$

This action makes $\mathbb{A}^1$ into a homogenous $\mathbb{G}_a$-variety. The stabilizer of $a = 0$ is the subgroup $\{0\}$. But it is not true that $\mathbb{G}_a/\{0\}$ is isomorphic to $\mathbb{A}^1$ via the map ϕ. (It happens to be isomorphic to $\mathbb{A}^1$ via a different map - the identity - but this is "a coincidence". One thus sees that if we are looking for a model for G/H using a homogeneous G-variety we need to be careful about issues of separability.]]

Step 2. In this step one constructs G/H as a ringed space. As a set G/H is the set of cosets G/H and $a = eH$. The function $\pi : G \to G/H$ is just $\pi(x) = \bar{x} = xH$. One gives G/H the quotient topology: $U \subset G/H$ is open iff $\pi^{-1}(U)$ is open in G. Thus, the function π is continuous and open.

Finally, for an open set $U \subset G/H$ define

$$\begin{aligned} \mathcal{O}(U) &= \{f : U \to k : f \circ \pi \text{ is regular on } \pi^{-1}(U) \\ &= \mathcal{O}_G(U)^H. \end{aligned} \tag{11.2.1}$$

This is a sheaf of rings on G/H.

Step 3. Show that the map of ringed spaces

$$(G/H, a) \to (X, x), \qquad gH \mapsto gx,$$

where (X, x) is from step 1 is an isomorphism. It thus shows G/H is a variety, in fact a quasi-projective variety. $\square$

We can see some consequences of the proof:

- $G \to G/H$ is open, surjective and its fibers are the cosets of H. $\dim(G/H) = \dim(G) - \dim(H)$.
- G/H is quasi-projective.
- if $H_1 \supseteq H$ there is a natural surjective open morphism $G/H \to G/H_1$.

Proposition 11.2.2. *A stronger universal property holds for the quotient G/H: Let $\phi : G \to Y$ be a morphism such that $\forall g \in G, h \in H, \phi(gh) = \phi(g)$. Then there exists a unique* *morphism* $\bar{\phi} : G/H \to Y$ *making the following diagram commutative*

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & Y \\ & G/H \arrow[ur, "\bar{\phi}"'] & \end{tikzcd}$$

Proof. Uniqueness is clear, as two morphism agreeing as functions are equal and $\bar{\phi}$ is determined as a function by the commutativity of the diagram. Define

$$\bar{\phi} : G/H \to Y, \qquad \bar{\phi}(\bar{x}) = \phi(x).$$

This is a well-defined function rendering the diagram commutative. It is a continuous function: let $U \subset Y$ be open, then $\bar{\phi}^{-1}(U) = \pi(\phi^{-1}(U))$. Since $\phi^{-1}(U)$ is open and π is open, also $\bar{\phi}^{-1}(U)$ is open. Thus, the only property we still need to show is that if $f \in k[U]$ is regular then $\bar{\phi}^* f \in k[\bar{\phi}^{-1}(U)]$ is regular. To show that we need to show that $\pi^*\bar{\phi}^* f \in k[\phi^{-1}(U)]$ is regular and H invariant. Now, $\pi^*\bar{\phi}^* = \phi^*$ and $\phi^* f(xh) = f(\phi(xh)) = f(\phi(x)) = \phi^* f(x)$. $\square$

Example 11.2.3. Recall that GL_n acts transitively on the flag variety $\mathscr{F}_{(d_1,\dots,d_t)}$ parameterizing a flag of subspaces $V_0 = \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_t \subset V = k^n$, where $\dim(V_i) = d_i$. Let $d_0 = 0$ and $d_{t+1} = n$. The stabilizer of the standard flag

$$\{V_i\} : \quad V_i = \mathrm{Span}_k\{e_1, \dots, e_{d_i}\}$$

is the subgroup

$$P = \left\{ \begin{pmatrix} A_1 & * & * & \dots & * \\ & A_2 & * & \dots & * \\ & & A_3 & \dots & * \\ & & & \ddots & \\ & & & & A_{t+1} \end{pmatrix} : A_i \in GL_{d_i - d_{i-1}}, i = 1, \dots, t+1 \right\},$$

where A_1 is a square matrix of size d_1, A_2 is a square matrix of size $d_2 - d_1$, A_3 is a square matrix of size $d_3 - d_2$ and so on, and A_{t+1} is of size $n - d_t$. The flag variety is projective and we have a natural bijective morphism

$$\mathrm{GL}_n/P \to \mathscr{F}_{(d_1,\dots,d_t)}.$$

In fact, this morphism is an isomorphism, but we shall not prove it here. It requires an understanding of the tangent space of $\mathscr{F}_{(d_1,\ldots,d_t)}$ at each of its point. The following Lemma, whose proof can be found in Springer, suffices to conclude many properties of GL_n/P.

Lemma 11.2.4. *Let $\phi : Y_1 \to Y_2$ be a bijective morphism of G-varieties, where G is a linear algebraic group. Then, for every variety Y, the morphism $\phi \times 1_Y : Y_1 \times Y \to Y_2 \times Y$ is (topologically) a homeomorphism.*

We mention one last important result about quotients.

Proposition 11.2.5. *Let G be a linear algebraic group and H a closed normal subgroup of G. Then the quotient G/H has a natural structure of an algebraic group. Further, G/H is an affine algebraic group whose coordinate ring is $k[G]^H$.*

Thus, for example, $\mathrm{PGL}_n = \mathrm{GL}_n/\mathbb{G}_m$ is an affine algebraic group.

12. Parabolic subgroups, Borel subgroups and solvable subgroups

12.1. **Complete varieties.** A variety X is called **complete** if for every variety Y the projection map $X \times Y \to Y$ is closed. Here are some fundamental facts about complete varieties.

- A quasi-projective variety is complete if and only if it is projective.
- A closed subvariety of a complete variety is complete.
- If X_1, X_2 are complete varieties so is $X_1 \times X_2$.
- If X is complete and irreducible then any regular function on X is constant.
- X is complete and affine if and only if X is finite.
- If $\phi\colon X \to Y$ is a morphism then $\phi(X)$ is closed and complete.

Of these facts the hardest to prove is the first one. The rest are not too hard to prove from the definition, or deduce from the first fact. According to these facts $\mathbb{A}^1$ is not complete and indeed we can show that directly. Consider the closed set $Z = \{(x,y) : xy = 1\}$ in $\mathbb{A}^1 \times \mathbb{A}^1$. Its projection to $\mathbb{A}^1$ is $\mathbb{A}^1 - \{0\}$, which is not closed.

12.2. **Parabolic subgroups.** A closed subgroup P of G is called **parabolic** if G/P is a complete variety.

Lemma 12.2.1. *If P is parabolic then G/P is a projective variety.*

Proof. We know already that G/P is a quasi-projective variety. Since it is also complete, it must be projective. □

Lemma 12.2.2. *Let G act transitively on a projective variety V and let $v_0 \in V$, $P = \mathrm{Stab}_G(v_0)$. Then P is parabolic.*

Proof. The natural map

$$G/P \to V, \qquad gP \mapsto gv_0,$$

is a bijective map of homogeneous G-varieties. By Lemma 11.2.4, for every Y the map,

$$G/P \times Y \to V \times Y,$$

is a homeomorphism and it commutes with the projection to Y. Thus, G/P is complete if and only if V is complete, which, being projective, it is. □

Corollary 12.2.3. *The subgroup P of* GL_n *given by*

$$P = \left\{ \begin{pmatrix} A_1 & * & * & \dots & * \\ & A_2 & * & \dots & * \\ & & A_3 & \dots & * \\ & & & \ddots & \\ & & & & A_{t+1} \end{pmatrix} : A_i \in GL_{d_i - d_{i-1}}, i = 1, \dots, t+1 \right\},$$

is a parabolic subgroup.

Corollary 12.2.4. *Let $\{e_1, \dots, e_n, \delta_1, \dots, \delta_n\}$ be the standard symplectic basis of k^{2n}, relative to the pairing $\langle x, y\rangle =^t xJy$, where $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. Thus, $\langle e_i, e_j\rangle = \langle \delta_i, \delta_j\rangle = 0$ for all i, j and $\langle e_i, \delta_j\rangle = -\langle \delta_j, e_i\rangle = \delta_{ij}$ (where δ_{ij} is Kronecker's delta).*

Let $d_0 = 0 < d_1 < \dots < d_t = n$ be integers and let $\mathscr{F}^{iso}_{d_1,\dots,d_t}$ be the projective variety parameterizing flags $\{0\} \subset V_1 \subset V_2 \subset \dots \subset V_t$, where each V_i is isotropic and of dimension d_i. By Witt's theorem, the symplectic group Sp_{2n} *acts transitively on $\mathscr{F}^{iso}_{d_1,\dots,d_t}$. The stabilizer of the standard flag $V_i = \mathrm{Span}\{e_1, \dots, e_{d_i}\}$ is thus a parabolic subgroup. This stabilizer has the shape*

$$\mathrm{Sp}_{2n} \cap \left\{ \begin{pmatrix} A_1 & * & * & \dots & * \\ & A_2 & * & \dots & * \\ & & A_3 & \dots & * \\ & & & \ddots & \\ & & & & A_{t+1} \end{pmatrix} : A_i \in GL_{d_i - d_{i-1}}, i = 1, \dots, t \right\},$$

where $d_{t+1} = 2n$ and A_{t+1} is in GL_n*. Now, it is easy to verify that a matrix $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ is symplectic iff $A \in \mathrm{GL}_n$, $D = {}^tA^{-1}$ and ${}^tBD$ is symmetric. Note then that D has the shape*

$$\left\{ \begin{pmatrix} A_1^{-1} & 0 & 0 & 0 & 0 \\ * & A_2^{-1} & 0 & 0 & 0 \\ * & * & A_3^{-1} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & A_t^{-1} \end{pmatrix} : A_i \in GL_{d_i - d_{i-1}}, i = 1, \dots, t \right\}.$$

Proposition 12.2.5. *Let $Q \subset P \subset G$ be closed subgroups. If Q is parabolic in G then it is parabolic in P.*

Proof. We have an injective map $P/Q \to G/Q$. The image is closed, as its complement is the image of the open set $G - P$ under the projection map $G \to G/Q$. Thus, the image is complete and in bijection by a map of homogeneous P-spaces with P/Q. It follows that P/Q is complete and so Q is parabolic in P. $\square$

Proposition 12.2.6. *If P is parabolic in G and Q is parabolic in P then Q is parabolic in G.*

Proof. We need to show that for every Y the morphism,

$$G/Q \times Y \to Y,$$

is closed.

Let $A \subseteq G \times Y$ be a closed set such that

$$(a, y) \in A \Rightarrow (aq, y) \in A, \quad \forall q \in Q.$$

We shall call such a set **Q-closed** (or Q-closed in $G \times Y$). Then, to show that the morphism $G \times Y \to Y$ is closed is equivalent to showing that the morphism $G \times Y \to Y$ takes a Q-closed set in $G \times Y$ to a closed set in Y.

Let A be a Q-closed set in $G \times Y$. Consider the morphism

$$P \times G \times Y \to G \times Y, \qquad (p, g, y) \mapsto (gp, y).$$

The preimage of A is

$$A^+ = \{(p, g, y) : (gp, y) \in A\}.$$

It is a Q-closed set in $P \times (G \times Y)$. Since Q is parabolic in P, the projection of A^+ to $G \times Y$ is thus closed. This projection is equal to the set

$$\tilde{A} = \{(gp, y) : (g, y) \in A, p \in P\}.$$

(Indeed, if $(g, y) \in A, p \in P$ then $(p^{-1}, gp, y) \in A^+$; conversely, if $(p, g, y) \in A^+$ then $(gp, y) \in A$ and so $(g, y) = ((gp)p^{-1}, y) \in \tilde{A}$.) The set $\tilde{A}$ is P-closed in $G \times Y$ and so its projection to Y is closed. But this projection is equal to the projection of A. $\square$

Proposition 12.2.7. *Let $P \subset Q \subset G$ be closed subgroups. If P is parabolic in G so is Q.*

Proof. There is a surjective morphism $G/P \to G/Q$. Since G/P is complete, so is its image, namely G/Q. $\square$

Proposition 12.2.8. *Let $P \subset G$ be a closed subgroup. Then P is parabolic in G if and only if P^0 is parabolic in G^0.*

Proof. First, G^0 is parabolic in G as G/G^0 is a finite group.

If P^0 is parabolic in G^0 then, because G^0 is parabolic in G, P^0 is parabolic in G by Proposition 12.2.6. Suppose that P is parabolic in G. Since P^0 is parabolic in P, it is parabolic in G, again by Proposition 12.2.6. Now $G \to G/P^0$ is an open map. Thus, the open set $G - G^0$ maps to an open set in G/P^0, whose complement is the closed set G^0/P^0. Since G^0/P^0 is a closed subset of a complete variety it is complete too and so P^0 is parabolic in G^0. $\square$

Theorem 12.2.9. *A connected linear group G contains a proper parabolic subgroup (that is, a parabolic subgroup different from G itself) if and only if G is non-solvable.*

Proof. Assume, without loss of generality, that G is a closed subgroup of GL_n. Recall also that the subgroup of upper triangular matrices in GL_n is solvable. (We have seen that the upper triangular unipotent matrices are a nilpotent, hence solvable group, and the quotient is $\mathbb{G}_m^n$ which is also solvable.)

The group GL_n, hence also G, acts on the projective space $\mathbb{P}^{n-1}$. Let X be a closed orbit for the action of G. Let $x \in X$ and $P = \mathrm{Stab}_G(x)$. Then the morphism of homogeneous G-varieties $G/P \to X, gP \mapsto gx$ is bijective. Since X is complete, so is G/P (we have made this argument before). Thus, P is parabolic. If $P = G$, it follows $k\tilde{x} \subset k^n$, where $\tilde{x}$ is a lift of x to k^n, is stable under G. Consider then the action of G on $\mathbb{P}(k^n/kx)$. Continuing this way, either we get a proper parabolic subgroup of G at some stage, or we arrive to a basis of k^n in which G is upper-triangular, hence solvable. Thus, if G is non-solvable, it has a proper parabolic subgroup.

Suppose now that G is solvable (and connected, by our original assumption). We need to show that it doesn't have any proper parabolic subgroups. We argue by induction on the dimension of G. The base case is dimension 0, where the statement is trivial as $G = \{1\}$.

Now, if G has proper parabolic group we may choose one of maximal dimension, say P. We may assume, using Proposition 12.2.8 that P is connected.

Recall that the commutator subgroup (G, G) of G is closed and connected (by an exercise). Consider the group

$$Q = P \cdot (G, G).$$

Since (G,G) is normal in G, $P\cdot(G,G)$ is a subgroup of G, equal to the subgroup generated by P and (G,G). Since both P and (G,G) are connected Q is connected. Since $Q \supseteq P$, Q is also parabolic. It follows that either (1) $Q = G$, or (2) $Q = P$.

Suppose that $Q = G$, then the homomorphism $(G,G) \to G/P$ is surjective and so induces a bijective morphism

$$\frac{(G,G)}{(G,G)\cap P} \longrightarrow \frac{G}{P}.$$

Since G/P is complete so is $(G,G)/((G,G)\cap P)$ and so $(G,G)\cap P$ is a parabolic subgroup of (G,G). Since G is solvable, (G,G) is a proper subgroup of G and so has strictly small dimension than G (G is irreducible) . Thus, by induction, $(G,G)\cap P$ is equal to (G,G). Then $(G,G)\subset P$ and $G = P\cdot(G,G) = P$, which is a contradiction.

The other case is $Q = P$. In this case $(G,G)\subset P$. In this case, because $G/(G,G)$ is commutative, P is a normal subgroup of G and so G/P is affine and complete, hence finite. But G/P is also connected. If follows that G/P is trivial and so that $G = P$, which is a contradiction. □

Theorem 12.2.10 (Borel's fixed point theorem)**.** *Let G be a connected linear solvable algebraic group and let X be a complete G-variety. Then G has a fixed point on X.*

Proof. Let $Y \subset X$ be a closed orbit and $y \in Y$. By the usual argument, the stabilizer of y is a parabolic subgroup, but cannot be a proper parabolic; hence it must be equal to G. That is, y is a fixed point for G. □

12.3. **Borel subgroups.** A closed subgroup B of a linear algebraic group G is called a **Borel subgroup** is it is closed, connected and solvable subgroup of G and is maximal relative to these properties.

Example 12.3.1. Let $G = \mathrm{GL}_n$ and B the upper triangular matrices. Then B is a Borel subgroup. Indeed, it is certainly a closed subgroup. It is solvable, as we had already remarked. It is also connected, being isomorphic, as a variety only, to $\mathbb{G}_m^n \times \mathbb{A}^{n(n-1)/2}$. Why is it maximal? Suppose that $B \subsetneqq P \subset G$ and P is connected. Then, since B is parabolic in G, B is parabolic in P and so P has a proper parabolic subgroup which implies that P is non-solvable.

Remark 12.3.2. The same argument shows that whenever we find a connected solvable closed subgroup of G which is parabolic, then it is a Borel subgroup. Thus, for example, the matrices of the form $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ such that A is an invertible $n \times n$ upper-triangular, $D = {}^tA^{-1}$, and B is a matrix such that ${}^tBD$ is symmetric, form a Borel subgroup of Sp_{2n}.

Theorem 12.3.3. *The following assertions hold true:*

1. *A closed subgroup of G is parabolic if and only if it contains a Borel subgroup.*
2. *A Borel subgroup is parabolic.*
3. *Any two Borel subgroups of G are conjugate.*

Corollary 12.3.4. *A closed subgroup of G is Borel if and only if it is a minimal parabolic subgroup.*

Proof of Theorem. Let P be a subgroup of G. Since P is parabolic in G if and only if P^0 is parabolic in G^0, and since any Borel subgroup is contained in G^0, we may assume that G is connected.

Let B be a Borel subgroup and P a parabolic subgroup. Then B acts on the complete variety G/P by $(b, gP) \mapsto bgP$. Since B is solvable and G/P complete, by Borel's fixed point theorem, B has a fixed point, say gP. Then $g^{-1}Bg \subset P$. Since $g^{-1}Bg$ is also a Borel subgroup, it follows that any parabolic subgroup contains a Borel subgroup. That prove the "only if" part of (1). If we prove (2) then we get that every Borel subgroup is parabolic and if $B \subset P \subset G$ also P is parabolic by Proposition 12.2.7, hence the "if" part of (1).

We now prove (2). Let B be a Borel. If G is solvable, then $G = B$ (we assumed G is connected) and a Borel is parabolic trivially. If G is not solvable, then there is a proper parabolic subgroup P of G. By what we had proven above, we may assume that $P \supset B$. Since the dimension of P is smaller than G's, by induction B is a parabolic subgroup of P. Thus, by Proposition 12.2.6, B is parabolic in G.

It remains to prove (3). Given two Borel subgroups B_1, B_2, since both are parabolic, we may conjugate B_1 into B_2 and B_2 into B_1. Thus, $\dim(B_1) = \dim(B_2)$. It follows that if $g^{-1}B_1g \subset B_2$ then, in fact, $g^{-1}B_1g = B_2$. □

Proposition 12.3.5. *Let $\phi : G \to H$ be a surjective homomorphism of algebraic groups and let P be a parabolic, resp. Borel, subgroup of G. Then $\phi(P)$ is parabolic, resp. Borel.*

Proof. Since any parabolic contains a Borel and every subgroup containing a parabolic is parabolic, it is enough to proof the assertion for Borel subgroups $P \subset G$. The subgroup $\phi(P)$ is closed, connected and solvable. Further, the morphism $G/P \to H/\phi(P)$ induced by ϕ is surjective and so $H/\phi(P)$ is complete. Thus, $\phi(P)$ is parabolic. As we have remarked above, this implies that $\phi(P)$ is Borel. $\square$

Proposition 12.3.6. *Let B be a Borel subgroup of a connected linear group G. Denote the centre of G and B by $C(G), C(B)$, respectively. Then*

$$C(G)^0 \subset C(B) \subset C(G).$$

Proof. The subgroup $C(G)^0$ is closed, connected and commutative, hence contained in some Borel subgroup. Since all Borel subgroups are conjugate, it is contained in every Borel subgroup.

Let $g \in C(B)$ and consider the morphism $G/B \to G$ given by $xB \mapsto gxg^{-1}x^{-1}$. Since G/B is projective (and so is its image) and G is affine, the image must consists of finitely many points. Since the image is connected, it must be just the identity element. That implies that $g \in C(G)$. $\square$

Remark 12.3.7. We shall later see that in fact $C(B) = C(G)$. This simplifies in practice the calculation of the centre of a group.

13. Connected solvable groups

We have skipped over many results in Chapter 5 of Springer, in order to be able to cover some deeper aspects of algebraic groups. In this chapter we will occasionally need such results. Those that have short proofs, will be proven; the proofs for the others can be found in the book.

13.1. **Lie-Kolchin.**

Theorem 13.1.1 (Lie-Kolchin)**.** *Let G be a closed connected solvable subgroup of GL_n. Then G can be conjugated into $\mathbb{T}_n$ - the upper triangular matrices.*

Proof. We use Borel's fixed point theorem. GL_n acts on $\mathbb{P}^{n-1}$. Thus G has a fixed point on $\mathbb{P}^{n-1}$, corresponding to a line kx_1. Consider now the action of G on $\mathbb{P}(k^n/kx_1)$. There is a fixed point, corresponding to a line in k^n/kx_1, hence to a plane kx_1+kx_2 in k^n. And so on. This way we obtain a basis $x_1, \dots, x_n$ in which G consists of upper triangular matrices.

Note that this proves that $\mathbb{T}_n$ is a Borel subgroup of GL_n. Conversely, supposing that $\mathbb{T}_n$ is a Borel of GL_n, one can also argue differently. Since G is connected and solvable, it is contained in a Borel subgroup of GL_n. Since all Borel subgroups are conjugate, it follows that G can be conjugated into $\mathbb{T}_n$. □

13.2. **Nilpotent groups.**

Lemma 13.2.1 (Springer's Corollary 5.4.8)**.** *Assume G to be a connected, nilpotent, linear algebraic group G. The set G_s of semi-simple elements is a subgroup of the centre of G.*

Proof. A commutator of length 1 is denoted $(x_1, x_2) = x_1x_2x_1^{-1}x_2^{-1}$; a commutator of length two is either $(x_1,(x_2,x_3))$ or $((x_1,x_2),x_3)$, etc.. Since G is nilpotent, there exists an n so that all iterated commutators of length n, $(x_1,(\dots(x_n,x_{n+1})\dots))$, are trivial.

Let $s \in G$ be a semisimple element, $\sigma = \mathrm{Int}(s)$ the automorphism of conjugation by s and $\chi\colon G \to G$ the morphism $x \mapsto \sigma(x)x^{-1}$ (and so $\chi(x) = (s,x)$). The commutators of length n being trivial implies that $\chi^n G = \{1\}$. One the other hand, one has the following identity on tangent spaces

$$d\chi_e : \mathfrak{g} \to \mathfrak{g}, \qquad d\chi_e = \mathrm{Ad}(s) - 1.$$

Since χ^n is the trivial map $\mathrm{Ad}(s)-1$ is nilpotent. On the other hand, $\mathrm{Ad}(s)$ is semisimple (any representation takes a semisimple element to a semisimple element) and the difference of two commuting semisimple operators is semisimple. Thus, $\mathrm{Ad}(s)-1$ is at the same time semisimple. It follows that $\mathrm{Ad}(s)-1=0$. Thus, $\mathrm{Ad}(s)$ is the identity map. We need to use the following

Lemma 13.2.2 (Springer's Corollary 5.4.5)**.** *Let G be a connected algebraic group. Let $s \in G$ be a semisimple element. Then:*

(1) *The conjugacy class $C = \{xsx^{-1} : x \in G\}$ is closed. The morphism $x \mapsto xsx^{-1}$ is separable.*
(2) *Let $Z = Z_G(s) = \{x \in G : xsx^{-1} = s\}$ be the centralizer of s. Then, with the notation $\mathfrak{Z} = T_{Z,e}$,*
$$\mathfrak{g} = \mathfrak{z} \oplus (\mathrm{Ad}(s)-1)\mathfrak{g}.$$

It follows that $\mathfrak{g} = \mathfrak{z}$ in our situation and so $\dim(G) = \dim(Z)$. Since Z is a closed subgroup of G and G is connected (hence irreducible), we must have $Z = G$. That is, s belongs to the centre of G.

Since the product of commuting semisimple elements is semisimple, we can now deduce that G_s is a group. □

The following proposition improves on Lemma 13.2.1.

Proposition 13.2.3. *Assume that G is a nilpotent connected algebraic group.*

(1) *The set G_s, resp. G_u, of semi-simple, resp. unipotent, elements is a closed, connected subgroup.*
(2) *G_s is a central torus of G.*
(3) *The product map $G_s \times G_u \to G$ is an isomorphism of algebraic groups.*

Proof. We have just seen that G_s is a subgroup of the centre of G. Let us assume that G is contained in GL_n. We may simultaneously diagonalize the elements of G_s, obtaining a decomposition of $V = V_1 \oplus \cdots \oplus V_d$ into subspaces according to characters of G_s; In particular, on each V_i each element of G_s acts by a scalar and because G_s is contained in the centre of G, each V_i is G-invariant. Let $G_i = G|_{V_i}$; the group G embeds into the product $G_1 \times \cdots \times G_d$. Note that each G_i is nilpotent and $(G_i)_s = (G_s)_i$ and so the semisimple elements of G_i are precisely the scalar matrices in G_i.

Each G_i is nilpotent, hence solvable. Thus, by Lie-Kolchin, we may assume that each G_i consists of upper-triangular matrices. The unipotent elements of G_i are precisely those with 1 on the diagonal, and thus, the Jordan decomposition associates to a matrix M in G_i a semisimple element M_s of G_i, which is therefore a diagonal matrix with the same diagonal entries as in M, as they have the same eigenvalues, and a unipotent part determined by $M = M_s M_u$. In fact, that semisimple part is a scalar matrix. One concludes that in this basis of V

$$G_s = G \cap \mathbb{D}_n, \qquad G_u = G \cap \mathbb{U}_n,$$

where we are denoting by $\mathbb{D}_n$ the diagonal matrices of GL_n and by $\mathbb{U}_n$ its upper triangular unipotent matrices. It follows that both G_s and G_u are closed subgroup. The map

$$G_s \times G_u \to G, \quad (g_s, g_u) \mapsto g_s g_u,$$

is a homomorphism because G_s is in the centre and it is bijective since it exhibits the Jordan decomposition for G. The inverse map is also a morphism, since, as we have seen, M_s is just the morphism $M \mapsto \mathrm{diag}(m_{11}, \dots, m_{nn})$ and $M_u = M_s^{-1} M$.

Finally, since G is connected this isomorphism shows that both G_s and G_u are connected.

□

13.3. **Solvable groups.**

Proposition 13.3.1. *Let G be a connected solvable algebraic group.*

(1) *The commutator subgroup (G, G) (also called the* **derived** *subgroup) of G is a closed connected unipotent normal subgroup.*
(2) *The set G_u of unipotent elements is a closed connected nilpotent normal subgroup of G. It contains (G, G). The quotient G/G_u is a torus.*

Example 13.3.2. The group (G, G) is contained in G_u but may be smaller than G_u. This is so in the case G is unipotent (thus, under a suitable embedding to GL_n, consists entirely of upper triangular unipotent matrices) since it is then nilpotent and the derived series must descend to the identity. For example, look at $G = \mathbb{U}_n$.

To illustrate the proposition, let $\mathbb{T}_n$ be the upper triangular matrices of GL_n, which is a Borel subgroup of GL_n. The commutator subgroup is $\mathbb{U}_n$, which is equal to the set of unipotent elements of $\mathbb{T}_n$ (but, as said, this need not be the case in general). The quotient $\mathbb{T}_n/\mathbb{U}_n$ is a torus. In fact, it is isomorphic to $\mathbb{D}_n$. As we'll see, the fact that there is a subgroup in $\mathbb{T}_n$ isomorphic to the quotient $\mathbb{T}_n/\mathbb{U}_n$ is not a coincidence.

Proof. We already know that (G, G) is a closed connected normal subgroup. To see it is unipotent, embed G in $\mathbb{T}_n$, for some n, by Lie-Kolchin. Then it is easy to check that (G, G) is a closed subgroup of $\mathbb{U}_n$, hence consists of unipotent elements.

As for (2), still by Lie-Kolchin, we view G as a closed subgroup of $\mathbb{T}_n$. Thus, $G_u = G \cap \mathbb{U}_n$ and so is a closed normal subgroup of G. We have an injective homomorphism of algebraic groups,

$$G/G_u \hookrightarrow \mathbb{T}_n/\mathbb{U}_n \cong \mathbb{D}_n.$$

Thus G/G_u is commutative and all its elements are semisimple. As we have seen, G/G_u can then be diagonalized and being connected must be a torus. (Here we are avoiding arguing that G/G_u is isomorphic to its image, which is true in fact.)

To show G_u is connected, consider its identity component G_u^0. G_u is normal in G and G_u^0 is a characteristic subgroup of G_u so it is normal in G. Consider then the group G/G_u^0. The group homomorphism $G \to G/G_u^0$ takes unipotent elements to unipotent elements. Thus G_u/G_u^0 are unipotent elements of G/G_u^0. Since G/G_u is a torus, these are all the unipotent elements. That is

$$(G/G_u^0)_u = G_u/G_u^0.$$

Thus, the algebraic group $H = G/G_u^0$ has the property that H_u is finite and H/H_u is commutative.

Lemma 13.3.3. *Let J be a connected algebraic group and N a finite normal subgroup of J then N is contained in the center of J.*

Proof of Lemma. Let $n \in N$. Consider the map $J \to N, g \mapsto gng^{-1}$. The image is connected and contains n. Thus, the image is $\{n\}$ and so n is a central element. □

Coming back to the main proof, we conclude that H_u is central and so we conclude that H is nilpotent (because it is so modulo a central subgroup) and connected. By Proposition 13.2.3, H_u is connected, hence trivial. Thus, $G_u = G_u^0$. □

Definition 13.3.4. Let G be a connected solvable algebraic group. We call a torus T in G **maximal** if its dimension is $\dim(G/G_u)$.

It is clear that such a torus is indeed maximal in the sense that it is not properly contained in any other torus, because any torus of G maps injectively into G/G_u. The converse is also true: every torus is contained in a maximal torus in the sense that its dimension is $\dim(G/G_u)$. We will prove that later.

Theorem 13.3.5. *Let G be a connected solvable algebraic group.*

1. *Let $s \in G$ be a semisimple element. Then s lies in a maximal torus. In particular, maximal tori exist (take $s = 1$, if you must).*
2. *The centralizer $Z_G(s)$ of a semisimple element $s \in G$ is connected.*
3. *Any two maximal tori of G are conjugate.*
4. *If T is a maximal torus then the morphism $T \times G_u \to G$ is an isomorphism of varieties.*

Proof. We first prove (4). Since G/G_u is a torus and the homomorphism $T \to G/G_u$ is injective (as $T \cap G_u = \{1\}$, the identity being the only element that is both unipotent and semisimple), it is also surjective, by dimension considerations. Thus, the morphism

$$T \times G_u \to G$$

is bijective as well. It is a morphism of $T \times G_u$-homogeneous varieties, where we give G the action $(t, u) * x = txu^{-1}$. Since a morphism of homogeneous varieties is separable if and only if it is separable at one point (Springer, Theorem 5.3.2), it is enough to check that it is separable at the identity elements.

We have

$$\mathscr{L}(T \times G_u) = \mathscr{L}(T) \oplus \mathscr{L}(G_u) \to \mathscr{L}(G), \qquad (X, Y) \mapsto X - Y. \tag{13.3.1}$$

(This is so because we are considering the morphism of $T \times G_u$-varieties taking (t, u) to teu^{-1}. For the natural multiplication morphism $T \times G_u \to G$ the map is $(X, Y) \mapsto X + Y$.)

Lemma 13.3.6. $\mathscr{L}(T) \cap \mathscr{L}(G_u) = \{0\}$ *(interesection inside $\mathscr{L}(G)$).*

Proof of Lemma. We show that the Lie algebra of a torus consists of semisimple elements and the Lie algebra of a unipotent group consists of nilpotent elements (viewed as derivations they are linear operators). Hence their intersection is just $\{0\}$.

For a torus, we may reduce to $\mathbb{G}_m^n$, whose Lie algebra, thought of as left-invariant derivations, is spanned by the invariant derivations $\{t_1\partial/\partial t_1, \dots, t_n\partial/\partial t_n\}$. As we have proven, the characters are a basis for $k[\mathbb{G}_m^n]$. The effect of $t_i\partial/\partial t_i$ on a character $t_1^{a_1} \cdots t_n^{a_n}$

is multiplication by a_i. That means that in the basis of characters each invariant derivation is already diagonalized and so is a semisimple operator.

For a unipotent group, using Lie-Kolchin, we may reduce to the case of $\mathbb{U}_n$. It is isomorphic as a variety (not as a group!) to $\mathbb{A}^{n(n-1)/2}$ and so $k[\mathbb{U}_n] = k[t_{ij} : 1 \leq i < j \leq n]$. A basis for the Lie algebra consists of the functions $\partial/\partial t_{ij}$, for $i < j$. In this way we view the tangent space of $\mathbb{U}_n$ as a subspace $\mathfrak{gl}_n$. The operation $\delta \mapsto *\delta$ clearly agrees with this identification. Now, recall that for GL_n we have

$$D_{ij} = *\partial/\partial t_{ij} = \sum_{a=1}^{n} t_{ai}\partial(\cdot)/\partial t_{aj}.$$

Since on $\mathbb{U}_n$ the functions $t_{ij} = 0$ for $i > j$ and are equal to 1 for $i = j$, the invariant derivation in $\mathscr{L}(\mathbb{U}_n)$ corresponding to $\partial/\partial t_{ij}$, for $i < j$, is

$$K_{ij} = \partial(\cdot)/\partial t_{ij} + \sum_{1\leq a<i}^{n} t_{ai}\partial(\cdot)/\partial t_{aj}.$$

To show each linear combination of such operators is nilpotent it is enough to show each K_{ij} is nilpotent and for that it is enough to show that it eventually reduces the degree of every monomial. It is enough to consider

$$k_{ij} = \sum_{1\leq a<i}^{n} t_{ai}\partial(\cdot)/\partial t_{aj}.$$

Now k_{ij} takes a homogeneous polynomial (e.g., a monomial) to either zero, or a homogeneous polynomial of the same degree. However, if we put a lexicographic order on the monomial where the letters are ordered $t_{12} < t_{13} < \cdots < t_{1n} < t_{23} < t_{24} < \dots$ then it decreases the order because

$$t_{ai}\frac{\partial}{\partial t_{aj}}\Big(\prod_{1\leq k<\ell\leq n} t_{k\ell}^{\alpha_{k\ell}}\Big) = \begin{cases} 0 & \alpha_{aj} = 0 \\ \alpha_{aj}t_{ai}\left(\frac{\prod_{1\leq k<\ell\leq n} t_{k\ell}^{\alpha_{k\ell}}}{t_{aj}}\right) & \alpha_{aj} > 0. \end{cases}$$

(So, either the monomial is killed, or the weight is shifted to a preceding letter.) □

Thus, the map (13.3.1) on tangent spaces is injective. Since both source and target groups are non-singular and of the same dimension, the map on tangent spaces is also surjective. And so we have a bijective separable morphism $T \times G_u \to G$. By Springer, *loc. cit.*, it is an isomorphism. This finishes the proof of (4).

The rest of the theorem is proven by induction on the dimension of G_u.

Case 1. $\dim(G_u) = 0$. In this case, since G_u is connected, $G_u = \{0\}$ and Proposition 13.3.1 gives that $G = G/G_u$ is a torus and the theorem holds trivially.

Case 2. $\dim(G_u) = 1$. This case is non-trivial and will take a while to prove. Using again the classification of connected algebraic groups of dimension 1, we know that $G_u \cong \mathbb{G}_a$. Fix an isomorphism,

$$\phi : \mathbb{G}_a \to G_u,$$

and use the notation ψ for the quotient map

$$\psi : G \to S := G/G_u.$$

The group S is a torus of dimension $\dim(G) - 1$. Consider the map

$$\mathbb{G}_a \to G_u, \qquad a \mapsto g\phi(a)g^{-1}.$$

Since $\mathrm{Aut}(\mathbb{G}_a) = \mathbb{G}_m$, there is a unique scalar $\alpha(g)$ such that

$$g\phi(a)g^{-1} = \phi(\alpha(g) \cdot a).$$

It is easy to see that α is a character of G that factors through G/G_u. Thus, abusing notation, there is a character α of S such that

$$g\phi(a)g^{-1} = \phi(\alpha(\psi(g)) \cdot a).$$

Claim. *If α is trivial then G is commutative.*

Proof of Claim. We first remark that this doesn't hold for abstract groups. For example, $\{\pm 1\}$ is a normal subgroup of the quaternion group Q_8 of 8 letters. The action of G on $\{\pm 1\}$ by conjugation is trivial and $Q_8/\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}^2$ is commutative. Still Q_8 is not commutative. However, in our case, consider the commutator map

$$G \times G \to G, \qquad (x, y) \mapsto [x, y].$$

Since α is trivial, it means that G_u is in the centre (for every $g \in G, a \in \mathbb{G}_a$, $g\phi(a) = \phi(\alpha(\psi(g)) \cdot a) \cdot g = \phi(a)g$). Thus, the pairing factors through $G/G_u \times G/G_u \to G$. In fact, this pairing has image in G_u because taking the values mod G_u we get the commutator pairing on a torus. Thus, we have a pairing

$$G/G_u \times G/G_u \to G_u, \qquad (\bar{g}, \bar{h}) \mapsto ghg^{-1}h^{-1}.$$

Now, this pairing is in fact a homomorphism in, say, the first argument: $[\bar{g}_1\bar{g}_2, \bar{h}] = g_1g_2hg_2^{-1}g_1^{-1}h^{-1} = g_1[g_2, h]hg_1^{-1}h^{-1} = [g_1, h][g_2, h]$. Thus, since G/G_u is a torus and G_u is unipotent, the map $\bar{g} \mapsto [\bar{g}, h]$ must be trivial for any $g \in G, h \in G$ and that means G is commutative.

Thus, we have established that if α is trivial then G is commutative. In this case we have proven that $G = G_s \times G_u$ and the assertions of the theorem are easy to check. (There is a unique maximal torus, equal to G_s.). Thus, we assume henceforth that α is not trivial.

Let $s \in G$ be a semisimple element and let $Z = Z_G(S)$ be its centralizer. By Lemma 13.2.2,

$$\mathfrak{g} = \mathfrak{z} \oplus (\mathrm{Ad}(s) - 1)\mathfrak{g}, \qquad \mathfrak{z} := \mathscr{L}(Z).$$

The relation,

$$\psi(sxs^{-1}x^{-1}) = 1,$$

(that follows from the commutativity of G/G_u) gives on the tangent spaces the relation

$$d\psi \circ (\mathrm{Ad}(s) - 1) = 0.$$

Now, the map

$$d\psi : T_{G,e} \to T_{G/G_u,e},$$

is not the zero map. In fact, from the construction of $\psi : G \to G/H$ in general, it follows that $d\psi$ is surjective with kernel $\mathscr{L}(H)$. In particular, $\ker(d\psi) = \mathscr{L}(G_u)$, which is one dimensional and contains $(\mathrm{Ad}(s) - 1)\mathfrak{g}$. Thus, $\dim(\mathfrak{z}) = \dim(Z^0) \geq \dim(G) - 1$.

Consider first the case where $\alpha(\psi s) \neq 1$. It is important later that this case occurs. Indeed, if g is such that $\alpha(\psi g) \neq 1$ then also $\alpha(\psi g_s) \neq 1$. In this case, using $g\phi(a)g^{-1} = \phi(\alpha(\psi(g)) \cdot a)$, we see that

$$Z_G(s) \cap G_u = \{1\}.$$

This shows that Z^0 must be a proper subgroup of G and hence it is a closed connected subgroup of G of dimension $\dim(G) - 1$ with $Z_u^0 = \{1\}$. We proved that the quotient of a connected solvable group by its unipotent elements is a torus. Thus, Z^0, having dimension $\dim(G/G_u)$ is a maximal torus and by (4) we have

$$G = Z^0 G_u.$$

We claim that in fact $Z = Z^0$. Let $g \in G$. Then $g = g_0 g_u$ with $g_0 \in Z^0$ and $g_u \in G_u$. Then, s commutes with g_0 and so commutes with g if and only if it commutes with g_u. But that would imply $\alpha(\psi s) = 1$, unless $g_u = 1$ (meaning $g_u = \phi(0)$). It follows that $Z = Z^0$. We have therefore shown (1) and (2) provided $\alpha(\psi s) \neq 1$.

Suppose then that $\alpha(\psi s) = 1$. This means that $G_u \subseteq Z = Z_G(s)$ and so $\mathscr{L}(G_u) \subseteq \mathfrak{z}$. But, the decomposition $\mathfrak{g} = \mathfrak{z} \oplus (\mathrm{Ad}(s) - 1)\mathfrak{g}$ and $(\mathrm{Ad}(s) - 1)\mathfrak{g} \subset \mathscr{L}(G_u)$ is only possible if $\mathrm{Ad}(s) - 1 = 0$. That means that $Z = G$ and so that s is an element of the centre of G. In this case (2) is trivial. To see (1), namely that s lies in a maximal torus, take any element s' such that $\alpha(\psi s') \neq 1$. We have then seen that $Z_G(s')$, which surely contains s, is a maximal torus.

We still need to prove (iii) (still in the case $\dim(G_u) = 1$). Let T be a maximal torus. If T' is a maximal torus, each of its elements is semi-simple, and from our discussion above it follows that there must be an element $t' \in T'$ such that

$$\alpha(\psi t') \neq 1, \qquad T' = Z_G(t').$$

Let us write

$$t' = t\phi(a), \qquad t \in T, a \in \mathbb{G}_a.$$

If $a = 0$ then $t' \in T$ and $T' = Z_G(t') \supseteq T$, whence $T = T'$. Assume $a \neq 0$. Let $b \in \mathbb{G}_a, b \neq 0$. Then

$$\begin{aligned} t \cdot \phi(a + ([\alpha(\psi t')]^{-1} - 1)b) &= t \cdot \phi(a) \cdot \phi([\alpha(\psi t')]^{-1}b) \cdot \phi(b^{-1}) \\ &= t \cdot \phi(a) \cdot (t'^{-1}\phi(b)t') \cdot \phi(b^{-1}) \\ &= \phi(b)t'\phi(b)^{-1}. \end{aligned}$$

The point is that we can choose $b \neq 0$ so that the left hand side is t. We have shown that t' can be conjugated to get t. Thus, the centralizer of t', which is T', can be conjugated to contain the centralizer of t, which, in turn, contains T. That is, $\phi(b)T'\phi(b)^{-1} \supseteq T$. Dimensions being the same, and T and T' being connected, we must have $\phi(b)T'\phi(b)^{-1} = T$.

This finishes the proof in the case $\dim(G_u) = 1$. For the induction step we shall need the following lemma (that we use without proof; its proof is independent of our proof of the theorem).

Lemma 13.3.7 (Springer's Lemma 6.3.4)**.** *Under the assumptions of the theorem, assume that G is not a torus. There exists a closed normal subgroup N of G, contained in G_u and isomorphic to $\mathbb{G}_a$.*

We now assume that $\dim(G_u) > 1$ and choose such a subgroup N. Let $\bar{G} = G/N$, a connected solvable group, and let $\bar{G}_u = G_u/N$. Note that $\dim(\bar{G}_u) = \dim(G_u) - 1$ and $\bar{G}_u = (\bar{G})_u$. We can apply induction to $\bar{G}$ then.

Let s be a semisimple element of G and $\bar{s}$ its image in $\bar{G}$. It is a semisimple element and so lies in a maximal torus $\bar{T}$ of $\bar{G}$. Recall that this means that $\dim(\bar{T}) = \dim(\bar{G}) - \dim(\bar{G}_u) = \dim G - \dim G_u$. Consider the preimage $\tilde{T}$ of $\bar{T}$ in G. Since $\tilde{T}_u = N$ is one dimensional, we may apply induction and conclude that s lies in a maximal torus of $\tilde{T}$. This torus has dimension $\dim \tilde{T} - \dim N = \dim \bar{T} = \dim G - \dim G_u$ and so is a maximal torus of G as well. Hence, (1).

Let T, T' be maximal tori of G. Their projections to $\bar{G}$ are maximal tori and thus for some $\bar{b}$, $\bar{b}\bar{T}\bar{b}^{-1} = \bar{T}'$. It follows that bTb^{-1} is a maximal torus of $\tilde{T}'$, the preimage of $\bar{T}'$. Thus, we reduced the assertion to maximal tori of $\tilde{T}'$. Since $\tilde{T}'$ is connected (being fibered over $\bar{T}'$ with fiber $\mathbb{G}_a$) and $\tilde{T}'_u$ is one-dimensional, the assertion follows from the case we have already proven. This proves (3).

Consider now assertion (2). We know that $Z_{\bar{G}}(\bar{s})$ is connected. Let $\tilde{Z}$ be the preimage in G. It is the subgroup of elements g of G such that $[s, g] \in N$. $\tilde{Z}$ is a connected variety(being a fibration of $Z_{\bar{G}}(\bar{s})$ with fiber $\mathbb{G}_a$). It is thus a connected solvable group. The centralizer of s in G is contained in $\tilde{Z}$ and is equal to the centralizer of s in $\tilde{Z}$. If s is central everything is easy. If s is not central and $\dim \tilde{Z} < \dim G$ we have by induction on $\dim G$ that the centralizer is connected. If s is not central and $\dim \tilde{Z} = \dim G$ then $G = Z^0 \cdot N$ (because $Z^0 \twoheadrightarrow \bar{G}$ implies $Z^0 \twoheadrightarrow \bar{G}$). Further, for dimension reasons, $Z^0 \cap N$ is finite. But a unipotent group doesn't have finite subgroups. Thus $Z^0 \cap N = \{1\}$. As before (cf. page 80), one concludes that $Z = Z^0$. Alternately, one can argue that $G = \coprod zZ^0 \cdot N$ over coset representatives for Z/Z^0 and use that G is connected. $\square$

Corollary 13.3.8. *Let G be a connected solvable group and $H \subset G$ a closed subgroup all whose elements are semisimple.*

(1) *H is contained in a maximal torus of G. In particular, a subtorus of G is contained in a maximal torus (and so the maximal tori are maximal with respect to inclusion as well).*
(2) *The centralizer $Z_G(H)$ is connected and coincides with the normalizer $N_G(H)$.*

Proof. Since $H \cap G_u = \{1\}$, the morphism $H \to G/G_u$ is injective and so H is commutative. In fact, a torus. If H is contained in the centre of G the assertions about the centralizer and normalizer are obvious. Let $s \in H$. Then s is contained in a maximal torus T. We claim that $T \supset H$, otherwise the homomorphism $T \times H \to G$ shows that T is not maximal. (In particular, we proved that every maximal torus contains the centre of the group.)

Otherwise, let $s \in H$ be an element which is not in the centre of G. Then $H \subseteq Z_G(s)$, which is a connected subgroup of G of smaller dimension. By induction on the dimension of G we conclude that H is contained in a maximal torus T of $Z_G(s)$. It follows from (1) of the theorem that $Z_G(s)$ contains a maximal torus of G, hence the dimension of the maximal tori of G and of $Z_G(s)$ are the same. Thus, H is contained in a maximal torus of G.

It remains to prove the assertion about $N_G(H)$. Let $x \in N_G(H)$ and $h \in H$. Then

$$xhx^{-1}h^{-1} \in H \cap (G, G) \subseteq H \cap G_u = \{1\},$$

and so $x \in Z_G(H)$. □

Example 13.3.9. All Borel subgroups are conjugate. Thus, to study Borel subgroups of GL_n it is enough to consider $\mathbb{T}_n$. Also, every maximal torus of GL_n is contained in a maximal connected closed subgroup, that is, in a Borel. Thus, every maximal torus of GL_n is conjugate to a torus of $\mathbb{T}_n$. All maximal tori in a Borel are conjugate. Thus, every maximal torus of GL_n is conjugate to $\mathbb{D}_n$.

The unipotent elements of $\mathbb{T}_n$ are $\mathbb{U}_n$, the upper unipotent matrices. The map

$$\mathbb{D}_n \times \mathbb{U}_n, \qquad (M, N) \mapsto MN,$$

is an isomorphism of varieties, but not of groups (unless $n = 1$) as $\mathbb{D}_n$ and $\mathbb{U}_n$ do not commute in $\mathbb{T}_n$ for $n > 1$. It is important to note that there are plenty of elements in $\mathbb{T}_n$ that are semi-simple besides $\mathbb{D}_n$. For example, any matrix whose diagonal entries are distinct is semisimple. Every semisimple element is contained in a maximal torus, in particular, every such element can be conjugated in $\mathbb{T}_n$ to a diagonal matrix, and vice-versa. That is, the semisimple elements of $\mathbb{T}_n$ are the conjugates of the diagonal matrices.

Let us take a diagonal matrix and consider its centralizer in $\mathbb{T}_n$. If the matrix is a scalar then the centralizer is obviously $\mathbb{T}_n$. Consider, as a sample case, a matrix of the form $\mathrm{diag}(a, 1, \dots, 1)$, where $a \notin \{0, 1\}$. One calculates that the centralizer are the matrices (t_{ij}) in $\mathbb{T}_n$ such that $t_{1j} = 0$ for $j > 1$. Thus, the centralizer is isomorphic to $\mathbb{G}_m \times \mathbb{T}_{n-1}$.

Example 13.3.10. Let us now look at Borel subgroups in Sp_{2n}. It will be convenient to change basis so that the pairing is given by

$$\langle x, y\rangle = {}^t x \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} y, \qquad K := \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}.$$

$(K = K^{-1}.)$ As we have done before, the symplectic group acts transitively on the flag variety of maximal isotropic flats. The stabilizer of the standard flag are the following matrices:

$$\left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} : {}^t\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}, A \in \mathbb{T}_n \right\}.$$

One finds that this amounts to the following conditions:

$$D = K^{-1} \cdot {}^tA^{-1} \cdot K, \quad A \in \mathbb{T}_n, \quad {}^tBKD \text{ is symmetric.}$$

Note that if $M = (m_{ij})$ and σ is permutation of $\{1, 2, \dots, n\}$,

$$\sigma = (1n)(2\ n-2)(3\ n-2)\cdots,$$

then

$$K^{-1}MK = (m_{\sigma(i)\sigma(j)}).$$

Let

$$^{\tau}A = K \cdot^t A^{-1} \cdot K.$$

Then $A \mapsto {}^{\tau}A$ is an automorphism that preserves $\mathbb{T}_n$ and furthermore, if $A = (a_{ij}) \in \mathbb{T}_n$ then the diagonal entries of ${}^{\tau}A$ are $a_{nn}^{-1}, \dots, a_{11}^{-1}$. This means that we can write the Borel we have found as

$$\mathbb{B} := \left\{ \begin{pmatrix} A & B \\ 0 & {}^{\tau}A \end{pmatrix} : A \in \mathbb{T}_n, {}^tBKD \text{ symmetric} \right\}.$$

Note that these are upper triangular matrices lying also in GL_{2n} and thus

$$\mathbb{B}_u = \left\{ \begin{pmatrix} A & B \\ 0 & {}^\tau A \end{pmatrix} \in \mathbb{B} : A \in \mathbb{U}_n \right\}.$$

We find that the torus

$$\left\{ \mathrm{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) : t_i \in k^\times \right\}$$

is a maximal torus since it maps isomorphically onto $\mathbb{B}/\mathbb{B}_u$.

As before, every maximal torus of Sp_{2n} is conjugate to this torus and every maximal torus of $\mathbb{B}$ is conjugate to this torus in $\mathbb{B}$. Every Borel of Sp_{2n} is conjugate to $\mathbb{B}$. Every semisimple element of $\mathbb{B}$ is conjugate to an element of the torus we found. As an example of how a centralizer in $\mathbb{B}$ of such an element may look like, consider a diagonal matrix $\mathrm{diag}(a, 1, \dots, 1, a^{-1})$. One calculates that the stablizer are the matrices of the form

$$\left(\begin{array}{cccc|ccc} a & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & & A & & & B & \\ \hline & & & & & {}^\tau A & 0 \\ & & & & & & a^{-1} \end{array}\right).$$

The middle matrix here is the Borel of Sp_{2n-2}.