ALGEBRAIC GROUPS: PART III

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Contents

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10. The Lie algebra of an algebraic group

10.1. **Derivations.** Let R be a commutative ring, A an R-algebra and M an A-module. A typical situation for us would be the case where R is an algebraically closed field, A the ring of regular functions of an affine k-variety and M is either A itself, or A/M, where M is a maximal ideal. Returning to the general case, define D, an **M-valued R-derivation of A**, to be a function

$$D : A \to M,$$

such that D is R-linear and

$$D(ab) = a \cdot D(b) + b \cdot D(a).$$

We have used the dot here to stress the module operation: $a \in A$ and $D(b) \in M$ and $a \cdot D(b)$ denotes the action of an element of the ring A on an element of the module M. The collection of all such derivations, $\mathrm{Der}_R(A, M)$, is an A-module, where we define

$$(f \cdot D)(a) = f \cdot D(a), \qquad f, a \in A.$$

Example 10.1.1. Let k be an algebraically closed field, X an affine k-variety, $\mathcal{O}_{X,x}$ the local ring of x on X and view k as an $\mathcal{O}_{X,x}$-module via $f \cdot \alpha = f(x)\alpha$, for $f \in \mathcal{O}_{X,x}, \alpha \in k$. Then $\mathrm{Der}_k(\mathcal{O}_{X,x}, k)$ are the k-linear functions

$$\delta : \mathcal{O}_{X,x} \to k,$$

such that $\delta(fg) = f \cdot \delta(g) + g \cdot \delta(f)$. Using the definition of the module structure, these are the functions $\delta : \mathcal{O}_{X,x} \to k$ such that

$$\delta(fg) = f(x)\delta(g) + g(x)\delta(f).$$

10.2. **The tangent space.** There are many interpretations for the tangent space. We bring here three such. Yet another one can be given using the notion of $k[\epsilon]$ points ($\epsilon^2 = 0$). See Springer's, or Hartshorne's book.

10.2.1. *An intrinsic algebraic definition.* Let X be a variety. The tangent space at $x \in X$, $T_{X,x}$, is

$$T_{X,x} := (\mathfrak{m}_x/\mathfrak{m}_x^2)^*,$$

namely, $T_{X,x} = \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$, where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$ and $\mathfrak{m}_x/\mathfrak{m}_x^2$ is viewed as a $k = \mathcal{O}_{X,x}/\mathfrak{m}_x$-vector space.

10.2.2. *A naive non-intrinsic geometric definition.* Assume that X is affine (else, pick a Zariski open affine negihborhood of x), say $X \subseteq \mathbb{A}^n$, defined by an ideal I, and let M be the maximal ideal of $k[X]$ comprising the functions vanishing at x. Then,

$$\mathcal{O}_{X,x} = k[X]_{\mathfrak{m}_x}, \qquad \mathfrak{m}_x = M\mathcal{O}_{X,x},$$

and so

$$\mathcal{O}_{X,x}/\mathfrak{m}_x = k[X]/M, \qquad \mathfrak{m}_x/\mathfrak{m}_x^2 = M/M^2.$$

As a result,

$$T_{X,x} = (M/M^2)^*.$$

The point is that $(M/M^2)^*$ affords a description which is closer to our geometric intuition. Suppose that $T_1, \dots, T_n$ are the variables on $\mathbb{A}^n$ and f is a function vanishing on X. Develop f into a Taylor series at $x = (x_1, \dots, x_n)$; the leading term, which we denote df_x, is

$$d_x f = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i}(x)(T_i - x_i). \tag{10.2.1}$$

We define $T_{X,x}^{\text{naive}}$ to be the affine-linear variety defined by all the equations (10.2.1) as f ranges over I:

$$d_x f = 0, \quad f \in I.$$

We remark that if $I = \langle f_1, \dots, f_m \rangle$ then it is enough to use $d_x f_1, \dots, d_x f_m$ to define $T_{X,x}^{\text{naive}}$. Thus, $T_{X,x}^{\text{naive}}$ with origin at x, is the solutions to the homogenous system of equations

$$\left(\frac{\partial f_i}{\partial T_j} \right)_{1 \leq i \leq m, 1 \leq j \leq n}.$$

We can in fact define $d_x f$ for any $f \in k[X]$, by the same formula (10.2.1), and view it as a function on $T_{X,x}^{\text{naive}}$, which is linear once we make $x = (x_1, \dots, x_n)$ the origin. In this interpretation $d_x f$, for $f \in I$, is the zero function on $T_{X,x}^{\text{naive}}$. Now, since $k[X] = k + M$ and $d_x f = 0$ if f is constant, we may view d_x as a map

$$d_x : M \to (T_{X,x}^{\text{naive}})^*.$$

Since $d_x(fg) = f(x)d_x(g) + g(x)d_x(f)$ we see that d_x vanishes on M^2 and so we get a map

$$d_x : M/M^2 \to (T_{X,x}^{\text{naive}})^*.$$

One proves this map is an isomorphism (it's not hard; see Springer, Humphries) and, thus, dualizing, we have

$$T_{X,x}^{\text{naive}} \cong (M/M^2)^* \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*,$$

showing that the naive non-intrinsic geometric definition agrees with the intrinsic algebraic definition.

10.2.3. *Via point derivations.* We can also view $T_{X,x}$, or, rather, $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ as point derivations at x. Let δ be a point derivation at x, $\delta : \mathcal{O}_{X,x} \to k$. Then δ is determined by its restriction to $\mathfrak{m}_x$ and $\delta|_{\mathfrak{m}_x^2} \equiv 0$. Thus, we get a map

$$\delta \mathfrak{m}_x/\mathfrak{m}_x^2 \to k,$$

which is k-linear. This is an element of $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. Conversely, a functional $\delta \in (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ defines a derivation δ by

$$\delta(f) = \delta(f - f(x) \pmod{\mathfrak{m}_x^2}).$$

10.3. **Regular points.** (Also called "non-singular", or "simple", points). Let X be an equi-dimensional variety (that is, all the components of X have the same dimension). A point $x \in X$ is called **regular** if

$$\dim_k T_{X,x} = \dim(X).$$

X itself is called regular if all its points are regular. A basic result is

Proposition 10.3.1. *The set of regular points in X is a dense Zariski-open set.*

Let $\varphi : X \to Y$ be a morphism$x \in X$, $y = \varphi(x)$. There is an induced homomorphism of local rings

$$\varphi^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x},$$

taking the maximal ideal $\mathfrak{m}_y$ into the maximal ideal $\mathfrak{m}_x$. There is therefore a k-linear map

$$\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2,$$

hence, by dualizing a k-linear map, which we denote $d\varphi_x$,

$$d\varphi_x : T_{X,x} \to T_{Y,y}.$$

There is another way to describe it. If δ is a point derivation at x then $d\varphi(\delta)$ is the point derivation at y defined by

$$d\varphi(\delta)(f) = \delta(f \circ \varphi).$$

The matching of these two definitions is an easy exercise. Other properties that follow easily are

$$d(\psi \circ \varphi)_x = d\psi_{\varphi(x)} \circ d\varphi_x, \qquad d(Id_X)_x = Id.$$

From this follows formally that if φ is an isomorphism so is $d\varphi_x$ for any x. The following theorem is useful to know (although we do not use it in the sequel).

Theorem 10.3.2. *Let $\varphi : X \to Y$ be a morphism. Then φ is an isomorphism if and only if it is bijective and $d\varphi_x$ is an isomorphism for all $x \in X$.*

Example 10.3.3. Consider the cuspidal curve $Y : y^2 = x^3$ in $\mathbb{A}^2$. The point $(0, 0)$ is a **singular point** (that is, it is not regular) as $M = (x, y, y^2 - x^3) = (x, y)$, $M^2 = (x^2, y_2, xy, y^2 - x^3) = (x^2, y^2, xy)$ and M/M^2 is *two* dimensional.

The normalization of Y, X is the affine line $\mathbb{A}^1$. The map

$$X \to Y, \qquad t \mapsto (t^2, t^3),$$

is a bijection. It cannon be an isomorphism though, because 0 is a regular point of X. And indeed, at the point 0 the map of tangent spaces can be calculated thus:

The point derivations of X at 0 are the derivations of the form $f(t) \mapsto \alpha \cdot (\partial f/\partial t)(0)$ where $\alpha \in k$. Let D be the derivation corresponding to $\alpha = 1$ (a basis for the 1-dimensional space of derivations). The point derivations of Y at $(0, 0)$ are the derivations $f(x, y) \mapsto \alpha \cdot (\partial f/\partial x)(0, 0) + \beta \cdot (\partial f/\partial y)(0, 0)$. The map $d\varphi_0$ is the following

$$d\varphi_0(D)(f(x, y)) = D(f(x, y) \circ \varphi) = D(f(t^2, t^3)) = 0,$$

as $f(t^2, t^3) - f(0, 0) \in (t^2)$.

Let us also calculate it the other way. The map $\mathcal{O}_{Y,(0,0)} \to \mathcal{O}_{X,0}$ is $f \mapsto f(t^2, t^3)$, which belongs to $\mathfrak{m}_0^2$ if $f \in \mathfrak{m}_{(0,0)}$. Once again, the map is the zero map.

Proposition 10.3.4. *Let X be an algebraic group. Then X is regular.*

Proof. Let $x_0 \in X$ be a regular point. Let $x_1 \in X$ an other point. The morphism

$$X \to X, \qquad x \mapsto x_1 x_0^{-1} x,$$

is an isomorphism taking x_0 to x_1, hence inducing an isomorphism

$$T_{X,x_0} \to T_{X,x_1}.$$

It follows that X is regular at x_1 as well. □

10.4. **Left invariant derivations.** Let G be an algebraic group over k. Let $A = k[G]$ and consider $\mathrm{Der}_k(A, A)$. If D_1, D_2 are derivations then so is

$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1,$$

as a simple calculation shows. This makes $\mathrm{Der}_k(A, A)$ into a **Lie algebra**. That means that apart from the vector space structure, the **bracket**

$$[\cdot, \cdot] : \mathrm{Der}_k(A, A) \times \mathrm{Der}_k(A, A) \to \mathrm{Der}_k(A, A),$$

is a k-bilinear alternating pairing, such that the Jacobi identity holds:

$$[D_1, [D_2, D_3]] + [D_2, [D_3, D_1]] + [D_3, [D_1, D_2]] = 0.$$

Recall that G acts on A locally finitely by

$$(\lambda_x f)(y) = f(x^{-1}y).$$

We define $\mathscr{L}(G)$, the **left invariants derivations of** G, as

$$\mathscr{L}(G) = \{D \in \mathrm{Der}_k(A, A) : D \circ \lambda_x = \lambda_x \circ D, \forall x \in G\}.$$

It is a Lie algebra under the bracket operation. Let

$$\mathfrak{g} = T_{G,e}$$

be the tangent space to G at the identity element e. We commonly think about it in terms of point derivations. There is a map

$$\mathrm{Der}_k(A, A) \to \mathfrak{g}, \qquad D \mapsto \{f \mapsto (Df)(e)\}.$$

This restricts to a k-linear map

$$\mathscr{L}(G) \to \mathfrak{g}.$$

In fact, a point of explanation is in order. The derivation D is defined on $k[G]$ and to define the point derivation we need to extend D to the local ring at e, $\mathcal{O}_{G,e} = k[G]_M$, where M is the maximal ideal corresponding to e. Firstly, if D can be extended to $\mathcal{O}_{G,e}$ then it would have to satisfy $D(f) = D(\frac{f}{g} \cdot g) = g \cdot D(\frac{f}{g}) + \frac{f}{g} \cdot D(g)$. That is, $D(\frac{f}{g}) = \frac{g \cdot D(f) - f \cdot D(g)}{g^2}$. This shows that if D extends to $\mathcal{O}_{G,e}$ this extension is unique. Secondly, $D(\frac{f}{g}) = \frac{g \cdot D(f) - f \cdot D(g)}{g^2}$ actually defines the extension of D to $\mathcal{O}_{G,e}$ (one has to verify it's well defined and is indeed a derivation, but this is just a simple verification).

Theorem 10.4.1. *The map*

$$\mathscr{L}(G) \to \mathfrak{g}$$

is an isomorphism of k-vector spaces. In particular, $\dim_k(\mathscr{L}(G)) = \dim(G)$.

Let $\varphi : G \to G'$ be a homomorphism of algebraic groups then the induced map,

$$d\varphi_e : \mathfrak{g} \to \mathfrak{g}',$$

is a homomorphism of Lie algebras (where $\mathfrak{g}, \mathfrak{g}'$ are given the bracket operation via the isomorphism to $\mathscr{L}(G), \mathscr{L}(G')$, respectively).

Remark 10.4.2. Although we do not need it in the proof, let's see that the map $\mathscr{L}(G) \to \mathfrak{g}$ is injective. Suppose that the point derivation

$$f \mapsto (Df)(e)$$

is identically zero. Then, for every $f \in k[G]$, and every $x \in G$,

$$\begin{aligned} 0 &= [D(\lambda_x f)](e) \\ &= [\lambda_x (Df)](e) \\ &= (Df)(x^{-1}). \end{aligned}$$

Thus, $Df = 0$ for all $f \in G$ and so D is the zero derivation.

Proof. Given a point derivation δ at e and $f \in A$, define the **convolution** of f with δ,

$$(f * \delta)(x) := \delta(\lambda_{x^{-1}} f).$$

Then, $f \mapsto f * \delta$ is a derivation:

$$\begin{aligned} ((fg) * \delta)(x) &= \delta(\lambda_{x^{-1}}(fg)) \\ &= \delta(\lambda_{x^{-1}} f \cdot \lambda_{x^{-1}} g) \\ &= \delta(\lambda_{x^{-1}} f) \cdot (\lambda_{x^{-1}} g)(e) + \delta(\lambda_{x^{-1}} g) \cdot (\lambda_{x^{-1}} f)(e) \\ &= \delta(\lambda_{x^{-1}} f) \cdot g(x) + \delta(\lambda_{x^{-1}} g) \cdot f(x) \\ &= (f * \delta)(x) \cdot g(x) + (g * \delta)(x) \cdot f(x) \\ &= ((f * \delta) \cdot g + (g * \delta) \cdot f)(x). \end{aligned}$$

It is left-invariant:

$$\begin{aligned}(\lambda_y(f*\delta))(x) &= (f*\delta)(y^{-1}x)\\ &= \delta(\lambda_{x^{-1}y}f)\\ &= \delta(\lambda_{x^{-1}}\lambda_y f)\\ &= ((\lambda_y f)*\delta)(x).\end{aligned}$$

We claim that this is an inverse to the map $\mathscr{L}(G)\to\mathfrak{g}$. Indeed, given $D\in\mathscr{L}(G)$, let δ be the derivation $f\mapsto (Df)(e)$. Then,

$$\begin{aligned}(f*\delta)(x) &= \delta(\lambda_{x^{-1}}f)\\ &= (D\lambda_{x^{-1}}f)(e)\\ &= (\lambda_{x^{-1}}Df)(e)\\ &= (Df)(x).\end{aligned}$$

Conversely, let $f\in k[G]$ and δ a point derivation at e. Then the derivation $f\mapsto (f*\delta)(e)$ is just δ as $(f*\delta)(e)=\delta(\lambda_e f)=\delta(f)$.

Now let $\varphi: G\to G'$ be a homomorphism. The map

$$d\varphi_e:\mathfrak{g}\to\mathfrak{g}'$$

is the map

$$d\varphi_e(\delta)(f') := \delta(\varphi^* f') = \delta(f'\circ\varphi),\qquad f'\in\mathcal{O}_{G',e'}.$$

Let $f=\varphi^*f'$. Let $\delta_1,\delta_2\in\mathfrak{g}$ (point derivations) and let $\delta_1' = d\varphi_e(\delta_1), \delta_2' = d\varphi_e(\delta_2)$. Then, on the one hand,

$$\begin{aligned}[\delta_1',\delta_2'](f') &= [*\delta_1', *\delta_2'](f')(e')\\ &= ((f'*\delta_2')*\delta_1')(e') - ((f'*\delta_1')*\delta_2')(e') \qquad (\textit{sic!})\\ &= \delta_1'(f'*\delta_2') - \delta_2'(f'*\delta_1')\\ &= \delta_1(\varphi^*(f'*\delta_2')) - \delta_2(\varphi^*(f'*\delta_1')).\end{aligned}$$

On the other hand,

$$\begin{aligned}d\varphi_e([\delta_1,\delta_2])(f') &= [\delta_1,\delta_2](\varphi^* f')\\ &= (((\varphi^*f')*\delta_2)*\delta_1)(e) - (((\varphi^*f')*\delta_1)*\delta_2)(e)\\ &= \delta_1((\varphi^*f')*\delta_2) - \delta_2((\varphi^*f')*\delta_1).\end{aligned}$$

It is therefore enough to prove the following identity,

$$(\varphi^* f') * \delta_2 = \varphi^*(f' * d\varphi_e(\delta_2)).$$

We do that by calculating the values of these functions on G at every $x \in G$. On the one hand,

$$\begin{aligned}((\varphi^* f') * \delta_2)(x) &= \delta_2(\lambda_x^{-1}\varphi^* f') \\ &= \delta_2(\varphi^*(\lambda_{\varphi(x)^{-1}} f')) \\ &= d\varphi_e(\delta_2)(\lambda_{\varphi(x)^{-1}} f').\end{aligned}$$

on the other hand,

$$\begin{aligned}\varphi^*(f' * d\varphi_e(\delta_2))(x) &= (f' * d\varphi_e(\delta_2))(\varphi(x)) \\ &= d\varphi_e(\delta_2)(\lambda_{\varphi(x)^{-1}} f').\end{aligned}$$

□

10.5. **Subgroups and Lie subalgebras.** Let H be a closed subgroup of G defined by an ideal J. The natural inclusion

$$H \to G,$$

induces a map on tangent spaces

$$T_{H,e} \to T_{G,e}.$$

The relations between local rings is $\mathcal{O}_{H,e} = \mathcal{O}_{G,e}/J$ and $\mathfrak{m}_{H,e} = \mathfrak{m}_{G,e}/J$. Thus

Theorem 10.5.1. *The image of $T_{H,e} \to T_{G,e}$ is the subspace of $T_{G,e}$ consisting of derivations δ such that $\delta(f) = 0, \forall f \in J$.*

This simple observation is very useful in calculating Lie algebras, as we shall see below. More theoretically, we have:

Theorem 10.5.2. *Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ the Lie algebra of H. Then $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and*

$$\mathfrak{h} = \{\delta \in \mathfrak{g} : f * \delta \in J, \forall f \in J\}.$$

Proof. Let $\delta \in \mathfrak{h}$ and $f \in J$. For $h \in H$ we have

$$(f * \delta)(h) = \delta(\lambda_{h^{-1}} f) = 0,$$

because $\lambda_{h^{-1}} f \in J$ and δ, being a derivation on (the localization of) $k[G]/J$, vanishes on J. Thus, $f * \delta \in J$.

Conversely, let $\delta \in \mathfrak{g}$ be a derivation such that $f * \delta \in J, \forall f \in J$. Then, $0 = (f * \delta)(e) = \delta(\lambda_{e^{-1}} f) = \delta(f)$. Thus, $\delta \in \mathfrak{h}$. □

10.6. **Examples.**

10.6.1. *The additive group* $\mathbb{G}_a$. Let the coordinate be t. The tangent space at zero is $k \cdot \frac{\partial(\cdot)}{\partial t}(0)$. Since the bracket operation is alternating, the bracket is trivial and this is the full story. What is the invariant derivation corresponding to $\delta := \frac{\partial(\cdot)}{\partial t}(0)$? Let f be a polynomial, $x \in \mathbb{G}_a$ and g the function $g(t) = f(t + x)$, then

$$\begin{aligned}(f * \delta)(x) &= \delta(\lambda_{-x} f)\\ &= \frac{\partial g}{\partial t}(0)\\ &= \frac{\partial f}{\partial t}(x),\end{aligned}$$

by the chain rule. That is, $*\delta$ is just the derivation $f \mapsto \frac{\partial f}{\partial t}$. To check, in general, that a derivation is invariant on $A = k[G]$, it is enough to test it on algebra generators, because, if $D\lambda_x f = \lambda_x Df$ and $D\lambda_x g = \lambda_x Dg$ then, $D\lambda_x(fg) = D(\lambda_x f \cdot \lambda_x g) = \lambda_x f D(\lambda_x g) + \lambda_x g D(\lambda_x f) = \lambda_x f \lambda_x(Dg) + \lambda_x g \lambda_x(Df) = \lambda_x D(fg)$, etc..

In the particular case at hand t is a generator and

$$\frac{\partial \lambda_{-x} t}{\partial t} = \frac{\partial(t + x)}{\partial t} = 1 = \lambda_{-x} \frac{\partial t}{\partial t}.$$

10.6.2. *The multiplicative group* $\mathbb{G}_m$. Again the tangent space is one dimensional and so the bracket is identically zero. We let δ be the point derivation at 1 given by $\delta(f) = \frac{\partial f}{\partial t}(1)$. For a fixed x, letting $g(t) = f(xt)$, the invariant derivation corresponding to δ is

$$\begin{aligned}(f * \delta)(x) &= \delta(\lambda_{x^{-1}} f)\\ &= \delta(g(t))\\ &= \frac{\partial g}{\partial t}(1)\\ &= x \cdot \frac{\partial f}{\partial t}(x).\end{aligned}$$

We conclude that up to multiplication by a scalar every invariant derivation on $\mathbb{G}_m$ is the derivation

$$f \mapsto t \cdot \frac{\partial f}{\partial t}.$$

We can again check. Call that derivation D. As an algebra $k[\mathbb{G}_m]$ is generated by $t^{\pm 1}$. We have

$$\lambda_{x^{-1}}Dt = \lambda_{x^{-1}}t = xt, \quad D(\lambda_{x^{-1}}t) = t \cdot \frac{\partial(tx)}{t} = xt,$$

and

$$\lambda_{x^{-1}}Dt^{-1} = \lambda_{x^{-1}}(t \cdot (-t^{-2})) = -1/(xt),$$
$$D(\lambda_{x^{-1}}t^{-1}) = t \cdot D(1/(xt)) = t \cdot (-x/(xt)^2) = -1/(xt).$$

10.6.3. *The general linear group* GL_n. A basis of the point derivations at e are the derivations

$$f \mapsto \frac{\partial f}{\partial t_{ij}}(1).$$

(The inclusion $\mathrm{GL}_n \hookrightarrow \mathbb{A}^{n^2}$ identifies the tangent spaces.) Let us calculate the corresponding invariant derivation. Let $\delta = \frac{\partial f}{\partial t_{ij}}(1)$. Consider the function $f((t_{ij})) = t_{k\ell}$, where k, ℓ are some fixed indices. Then, the function $g((t_{ab})) = \lambda_{(x_{ab})^{-1}} f((t_{ab}))$ is simply the function $(t_{ab}) \mapsto \sum_r x_{kr}t_{r\ell}$ and

$$\begin{aligned}(f * \delta)((x_{ab})) &= \delta(g((t_{ab}))) \\ &= \frac{\partial(\sum_r x_{kr}t_{r\ell})}{\partial t_{ij}}(1) \\ &= \delta_{j\ell} \cdot x_{ki}\end{aligned}$$

(here $\delta_{j\ell}$ is, unfortunately, the Kronecker delta symbol). Consider the derivations

$$D_{ij}f := \sum_a t_{ai}\frac{\partial f}{\partial t_{aj}}.$$

Since $f \mapsto \partial f/\partial t_{ab}$ is a derivation, and the derivations are a module over the coordinate ring, these are indeed derivations. The value of D_{ij} on the function $f((t_{ij})) = t_{k\ell}$ is $\delta_{j\ell} \cdot t_{ki}$. Since any derivation is determined by its values on the functions $t_{k\ell}$ it follows that D must be the derivation $*\delta$ and, in particular, left invariant.

If we write an element of $T_{\mathrm{GL}_n,1}$ as (m_{ij}), corresponding to the derivation $\sum m_{ij}\frac{\partial(\cdot)}{\partial t_{ij}}(1)$ and so to the left-invariant derivation $\sum_{ij} m_{ij}D_{ij}$ we can calculate the Lie bracket. It will be determined uniquely by taking the elementary matrices $E_{k\ell}, E_{ij}$, corresponding to $D_{k\ell}$

and D_{ij}. One calculates (it is enough to check on the basic functions we used before) that

$$[D_{kl}, D_{ij}] = \delta_{\ell i} D_{kj} - \delta_{jk} D_{il},$$

which is the derivation associated with $[E_{k\ell}, E_{ij}]$. Thus, we conclude that $\mathfrak{gl}_n$ is canonically identified with the k-vector space of all $n \times n$ matrices with Lie bracket

$$[X, Y] = XY - YX.$$

10.6.4. *Subgroups of* GL_n. Let H be a subgroup of GL_n defined by the vanishing of the ideal J, where we view $\mathrm{GL}_n \subset \mathbb{A}^{n^2}$. Then, the tangent space of GL_n is $\mathbb{A}^{n^2}$, identified as a Lie algebra with $M_n(k)$ with the bracket $XY - YX$ (see above), and the tangent space of H is defined as the subspace of $M_n(k)$ determined by the vanishing of the linear equations

$$df_e = \sum_{ij} \frac{\partial f}{\partial t_{ij}}(\mathrm{Id}_n), \qquad f \in J.$$

We note that if we develop such f as $f(\mathrm{Id}_n + (t_{ij})) = \sum_{ij} t_{ij} \frac{\partial f}{\partial t_{ij}}(0_n) + h.o.t.$ then $\mathfrak{h}$ is defined by the equations $\sum_{ij} t_{ij} \frac{\partial f}{\partial t_{ij}}(0_n)$, that is, by the equations

$$\sum_{ij} t_{ij} \frac{\partial f}{\partial t_{ij}} \quad (\mathrm{mod}\ (t_{ij} t_{k\ell})_{i,j,k,\ell}).$$

Consider for example $H = \mathrm{SL}_n$. Then H is defined by the equation $f(\mathrm{Id}_n + (t_{ij})) - 1 = \det(\mathrm{Id}_n + (t_{ij})) - 1 = 0$. Modulo squares of variables this is the equation

$$\sum t_{ii} = 0,$$

and we conclude that

$$\mathfrak{sl}_n = \{M \in M_n(k) : \mathrm{Tr}(M) = 0\}.$$

Consider the case of a bilinear form represented by a symmetric matrix $B = (b_{ij})$ (so that $\langle x, y \rangle = {}^t x B y$. The orthogonal group associated to it is

$$O_B = \{M \in \mathrm{GL}_n : {}^t M B M = B\}.$$

Write $M = \mathrm{Id}_n + (t_{ij})$ then, modulo squares, we have

$$\begin{aligned}(\mathrm{Id}_n + {}^t(t_{ij}))B(\mathrm{Id}_n + (t_{ij}) - B &= B + {}^t(t_{ij})B + B(t_{ij}) + (t_{ij})B(t_{ij}) - B \\ &= {}^t(t_{ij})B + B(t_{ij}).\end{aligned}$$

That is, the Lie algebra are the "B-skew-symmetric matrices",

$$\mathfrak{o}_B = \{M \in M_n(k) : {}^tMB = -BM\}.$$

Now $\mathrm{Tr}(M) = \mathrm{Tr}(BMB^{-1}) = \mathrm{Tr}(-{}^tMBB^{-1}) = -\mathrm{Tr}(M^t) = -\mathrm{Tr}(M)$ and so $\mathrm{Tr}(M) = 0$. It follows that

$$\mathfrak{so}_B = \mathfrak{o}_B.$$

This of course can be proven with no calculation. SO_B is of index 2 in O_B and is equal in fact to the identity component of O_B, hence they have the same Lie algebra.

In particular, for $B = \mathrm{Id}_n$ (corresponding to the quadratic form $q(x) = \sum x_i^2$), we have

$$\mathfrak{o}_n = \mathfrak{so}_n = \{M \in M_n(k) : {}^tM = -M\}.$$

10.6.5. *A useful observation.* One consequence of the fact that an algebraic group G oveer k is non-singular and that

$$\dim_k \mathscr{L}(G) = \dim_k \mathfrak{g} = \dim G,$$

is that we can calculate the dimension of G by calculating the dimension of its Lie algebra. Thus, we easily find that $\dim(\mathrm{GL}_n) = n^2$ and $\dim \mathrm{SL}_n = n^2 - 1$, which we knew of course, but also that $\dim \mathcal{O}_B = \dim S\mathcal{O}_B = \frac{1}{2}n(n-1)$.

10.6.6. *Products.* Let G_1, G_2 be algebraic groups. Then

$$\mathscr{L}(G_1 \times G_2) \cong \mathscr{L}(G_1) \oplus \mathscr{L}(G_2).$$

We leave that as an exercise.

10.6.7. *Tori.* Let T be a torus. There is a canonical isomorphism

$$\mathscr{L}(T) = k \otimes_{\mathbb{Z}} X_*(T).$$

Further, the bracket operation is trivial. We leave that as an exercise.

10.7. **The adjoint representation.** Let $A = k[G]$. We are going to define two actions of G on $\mathfrak{g}$ that will be shown to be equal. First, we note the action of G on $\mathscr{L}(G)$,

$$D \mapsto \rho_x \circ D \circ \rho_x^{-1},$$

where $x \in G$ and $(\rho_x f)(y) = f(yx)$. It is easy to check this is an action, using the $\lambda_y \rho_x = \rho_x \lambda_y$.

This induces an action on $\mathfrak{g}$ via the isomorphism $\mathfrak{g} \to \mathscr{L}(G)$; the action of x takes a derivation δ to a derivation μ such that

$$*\mu = \rho_x \circ *\delta \circ \rho_x^{-1}.$$

On the other hand, consider the action of G on $T_{G,e}$ coming from conjugation. Let $\mathrm{Int}(x)$ be the automorphism of G given by

$$\mathrm{Int}(x)(y) = xyx^{-1}.$$

We denote its differential at the identity by $\mathrm{Ad}(x)$. Thus,

$$\mathrm{Ad}(x) = d\,\mathrm{Int(x)}_e.$$

Note that, by definition,

$$\mathrm{Ad}(x)(\delta)(f) = \delta(f \circ \mathrm{Int}(x)) = \delta(\lambda_{x^{-1}}\rho_{x^{-1}} f).$$

We claim that $\mu = \mathrm{Ad}(x)(\delta)$. We may pass to $\mathfrak{g}$. Then $\mu(f)(e) = [\rho_x((\rho_x^{-1} f) * \delta)](e)$, which is equal to $[(\rho_x^{-1} f) * \delta](x) = \delta(\lambda_{x^{-1}}\rho_{x^{-1}} f) = \mathrm{Ad}(x)(\delta)(f)$. Thus, we have proven the following lemma.

Lemma 10.7.1. $\mathrm{Ad}(x)(\delta) = \rho_x \circ *\delta \circ \rho_x^{-1}$.

Let us consider now the case $G = \mathrm{GL}_n$. The map $\mathrm{Int}(x)$ is in fact a linear map on $\mathbb{A}^{n^2}$ and, as is well-known, the differential of a linear map is equal to the map. Thus:

Lemma 10.7.2. *Let $\delta \in \mathfrak{gl}_n = M_n(k)$ then*

$$\mathrm{Ad}(x)(\delta) = x\delta x^{-1}.$$

Corollary 10.7.3. *The adjoint representation $\mathrm{Ad} : \mathrm{GL}_n \to \mathrm{GL}(\mathfrak{gl}_n) \cong \mathrm{GL}_{n^2}$ is an algebraic representation.*

For a general algebraic group we cannot multiply elements of the group with elements of the Lie algebra, so there is no intrinsic formula like in the lemma. Yet, every algebraic group is isomorphic to a closed subgroup of GL_n and for those we have the following lemma.

Lemma 10.7.4. *Let H be a closed subgroup of GL_n and view $\mathfrak{h}$ as a Lie subalgebra of $\mathfrak{gl}_n$. Let $h \in H$ then $\mathrm{Ad}(h)(\delta) = h\delta h^{-1}$.*

Proof. This is just the statement that the differential of conjugation by h on H is the restriction of the differential of conjugation by h on GL_n. This is clear for example from the interpretation of the tangent space usingthe $\mathfrak{m}/\mathfrak{m}^2$ description. □

Corollary 10.7.5. *Let H be an algebraic group. The adjoint representation* $\mathrm{Ad} : H \to \mathrm{GL}(\mathfrak{h}) \cong \mathrm{GL}_n$ *is an algebraic representation ($n = \dim(H)$).*

This corollary is very important as it gives us a way to associate a canonical representation to an algebraic group.

10.7.1. $\mathfrak{ad}$ - *The differential of* Ad. We have constructed a linear representation

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}) \cong \mathrm{GL}_d(k),$$

where $d = \dim(G)$. Note though that $\mathrm{Ad}(x)$ has the additional property that it respects the Lie bracket on $\mathfrak{g}$. At any rate, we have an induced map of Lie algebras - the differential of Ad at the identity:

$$\mathfrak{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}).$$

The calculation for GL_n is given by the following theorem.

Proposition 10.7.6. *The homomorphism of Lie algebras*

$$\mathfrak{ad} : \mathfrak{gl}_n \to \mathfrak{gl}(\mathfrak{gl}_n) \cong M_{n^2},$$

is given by

$$\mathfrak{ad}(X)(Y) = [X, Y] = XY - YX.$$

The proof is a calculation that can either be done directly, or by more sophisticated arguments. In any case we omit it. It can be found in Springer's or Humphreys books.

By embedding an algebraic group G into GL_n we conclude:

Corollary 10.7.7. *Let G be an algebraic group. The homomorphism*

$$\mathfrak{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is given by

$$\mathfrak{ad}(X)(Y) = [X, Y] = X \circ Y - Y \circ X.$$