ALGEBRAIC GROUPS: PART II

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Contents

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4. Constructible sets

Let X be a quasi-projective variety, equipped with its Zariski topology. Call a subset C of X **locally closed** if $C = V \cap Z$, where V is open and Z is closed. We call a subset of X **constructible** if it is a finite disjoint union of locally closed sets.

Example 4.0.1. The set $T = (\mathbb{A}^2 - \{x = 0\}) \cup \{(0,0)\}$, being dense in $\mathbb{A}^2$ is not locally closed (Z would have to contain $\mathbb{A}^2 - \{x = 0\}$), hence would be equal to $\mathbb{A}^2$ and T is not open). But T is a constructible set, being the disjoint union of two locally closed sets $\mathbb{A}^2 - \{x = 0\}$ (open) with $\{(0,0)\}$ (closed).

Lemma 4.0.2. *The following properties hold:*

(1) *An open set is constructible.*
(2) *A finite intersection of constructible sets is constructible.*
(3) *The complement of a constructible set is constructible.*

Proof. If C is open then $C = C \cap X$ is the intersection of an open set with the closed set X. Similarly if C is closed.

To show part (2), suppose that $\coprod C_i$ and $\coprod D_i$ are constructible, where each C_i, D_j is locally closed. Since $\coprod C_i \cap \coprod D_j = \coprod C_i \cap D_j$, it is enough to prove that the intersection of two locally closed subsets $C = V_1 \cap Z_1$, $D = V_2 \cap Z_2$ is constructible. In fact, $C \cap D = (V_1 \cap V_2) \cap (Z_1 \cap Z_2)$ is even locally closed.

For part (3), the formula $(\coprod C_i)^c = \cap (C_i)^c$ and part (2), show that it is enough to prove that a complement of a locally closed set is constructible. Indeed, if $C = V \cap Z$ then $C^c = V^c \cup Z^c = Z^c \coprod (V^c - Z^c)$, a disjoint union of an open set with a closed set, both constructible as we have proven above. $\square$

Corollary 4.0.3. *A finite union of constructible sets is constructible. Thus, one may also define a constructible set as a union (not necessarily disjoint) of locally closed sets. The difference of two constructible sets is constructible.*

Proof. If C_i constructible, to show $\cup C_i$ is constructible, it is enough to show that $(\cup C_i)^c = \cap C_i^c$ is constructible. But that follows (2) and (3) of the Lemma. The proof for the difference is similar. $\square$

Corollary 4.0.4. *The collection of constructible sets is the smallest collection $\mathscr{F}$ of subsets of X containing all open sets and closed under finite intersections and complements.*

Proof. The lemma tells us that $\mathscr{F}$ is also closed under unions and is contained in the collection of constructible sets. To show the converse, it is enough to show that every locally closed set belongs to $\mathscr{F}$. This is true because $\mathscr{F}$ contains the open sets, hence the closed sets and hence their intersections. □

Exercise 3. *Every constructible set contains on open dense set of its closure.*

The main relevance of constructible sets to algebraic geometry is the following important theorem.

Theorem 4.0.5. *Let $\varphi : X \to Y$ be a morphism of varieties. Then φ maps constructible sets to constructible sets; in particular $\varphi(X)$ is constructible and contains a set open and dense in its closure.*

Corollary 4.0.6. *If φ is dominant, i.e., if the closure of $\varphi(X)$ is Y, then the image of φ contains and open dense set of Y.*

For the proof see Humphreys, p. 33, or, in more generality, Hartshorne, Algebraic Geometry (GTM 52), exercise 3.19 on page 94.

Exercise 4. *Find a morphism $\mathbb{A}^2 \to \mathbb{A}^2$ whose image is the set T of Example 4.0.1.*

5. Connected components

Let G be an algebraic group over an algebraically closed field k. There is no need to assume in this section that G is linear.

Proposition 5.0.7. *There is a unique irreducible component G^0 of G that contains the identity element e. It is a closed subgroup of finite index. G^0 is also the unique connected component of G that contains e and is contained in any closed subgroup of G of finite index.*

Proof. Let X, Y be irreducible components containing e, then $XY = \mu(X \times Y)$, $\overline{XY}$, X^{-1} and tXt^{-1}, for any $t \in G$, are irreducible and contain e. Since an irreducible component is, by definition, a maximal irreducible closed set, it follows that $X = XY = \overline{XY}$, $Y = XY$

and, so, $X = Y$. Since X^{-1} is also an irreducible component containing e, $X = X^{-1}$, and since tXt^{-1} is an irreducible component containing e, $X = tXt^{-1}$. Putting everything together, we conclude: (i) there is a unique irreducible component through e; (ii) it is closed; (iii) it is a subgroup; (iv) it is normal.

An irreducible component is connected. On the other hand, we have $G = \coprod_x xG^0$, the union being over coset representatives, each being an irreducible component. Thus, there must be finitely many of them, each is also open, and the union is a disjoint union of topological spaces. It follows that G^0 is a connected component of G, and in fact, the connected components equal the irreducible components.

Finally, given a subgroup H of G of finite index, we find that H^0 is of finite index in H and so in G, and is connected. Thus, H^0 is contained in G^0 and has finite index in it. Since G^0 is connected, we must have $H^0 = G^0$. $\square$

Example 5.0.8. The group $\mathbb{G}_a$ and $\mathbb{G}_m$ are connected. Hence, $\mathbb{G}_a^n$ and tori are connected. The unipotent group of GL_n is connected, being isomorphic to $\mathbb{A}^m$, for $m = n(n-1)/2$, and so the Borel subgroup is connected as well. The group GL_n is connected: it is the closed subset of $\mathbb{A}^{n^2} \times \mathbb{A}^1$ defined by $y\det(x_{ij}) - 1$, which is an irreducible polynomial. One can also argue that GL_n is irreducible, being an open non-empty subset of the irreducible space $\mathbb{A}^{n^2}$. It follows that the unitary groups $U(p,q)$ are connected.

Assume that B is a non-degenerate quadratic form over a field of characteristic different from 2. Let q be the associated quadratic form. The orthgonal group $\mathcal{O}_q$ is not connected, as it has a surjective homomorphism $\det : \mathcal{O}_q \to \{\pm 1\}$.

The spin groups $\mathrm{Spin}(V,q)$ are connected and so it follows that the groups $S\mathcal{O}_q$ are connected (and that $\mathcal{O}_q$ has two connected components). We give the argument under the assumption that q is not a square. Let Z be the closed subset of isotropic vectors in (V,q). Its complement U is connected, being an open dense set. In fact, it's an irreducible affine variety. Then, also the variety $U' = \{(u,t) : t^2 - q(u) = 0\}$ in $V \times \mathbb{A}^1$ is an irreducible affine variety. Further, the closed subvariety $W = \{v \in V : q(v) = 1\}$ is also irreducible because we have a surjective morphism $U' \to W, (v,t) \mapsto v/t$.

For every r there is a map

$$W^{2r} \to \mathrm{Spin}(V,q), \qquad (w_1, \dots, w_{2r}) \mapsto w_1 w_2 \cdots w_{2r}.$$

Let the image of these maps be S^{2r}. Then S^{2r} is irreducible, $S^{2r} \subseteq S^{2r+2}$ and every element of $\mathrm{Spin}(V, q)$ belongs to S^{2r} for some r. It follows that for some $r \gg 0$ we have $S^{2r} = \mathrm{Spin}(V, q)$ proving $\mathrm{Spin}(V, q)$ is irreducible, equivalently, connected.[1]

6. Subgroups

There is no need to assume in this section that G is linear.

Lemma 6.0.9. *Let U, V be two dense open subsets of an algebraic group G then $G = UV$.*

Proof. Let $g \in G$ and consider gV^{-1}. Since V^{-1} is open ($x \mapsto x^{-1}$ is a homeomorphism) also gV^{-1} is open and so $gV^{-1} \cap U \neq \emptyset$. Thus, for some $v \in V, u \in U$ we have $gv^{-1} = u$, which implies $g \in UV$. □

Proposition 6.0.10. *Let H be a subgroup of an algebraic group G. Let $\bar{H}$ be its closure. Then $\bar{H}$ is a closed subgroup of G. If H is constructible then $\bar{H} = H$.*

Proof. We first need to show that $\bar{H}$ is a subgroup. We have $\bar{H}^{-1} = \overline{H^{-1}} = \bar{H}$ and so it is closed under inverses. Since $HH \subset H$ we have $HH \subset \bar{H}$ and so for every $h \in H$ we have $\overline{hH} = h\overline{H} \subset \bar{H}$. That is, $H\bar{H} \subset \bar{H}$. Now, for every $h \in \bar{H}$ we have $\overline{Hh} \subset \bar{H}$ and so $\bar{H}h \subset \bar{H}$ and it follows that $\bar{H} \cdot \bar{H} \subset \bar{H}$ which shows that $\bar{H}$ is closed under multiplication.

Consider $\bar{H}$ as an algebraic group by itself. Since H is constructible it contains a subset U which is open and dense in $\bar{H}$. By the lemma $\bar{H} = UU \subset H$ and so $H = \bar{H}$. □

Proposition 6.0.11. *Let $\varphi : G \to J$ be a morphism of algebraic groups. Then the kernel of φ and the image of φ are closed subgroups. Furthermore, $\varphi(G^0) = \varphi(G)^0$.*

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[1] Had we wanted to use the Cartan-Dieudonné theorem we could have been more precise here. It tells us that every element of $S\mathcal{O}_q$ is a product of $q - p(q)$ reflections, where $p(q) = 0$ if the dimension is even and otherwise $p(q) = 1$. Thus, up to ± 1 we get every element of the spin group from $S^{q-p(q)}$. Thus, every element of the spin group is gotten from $S^{q+2-p(q)}$.

Proof. The kernel is closed being the preimage of a closed set. The image is a constructible subgroup of J, hence closed. We note that $\varphi(G^0)$ is connected, closed, and of finite index in $\varphi(G)$, hence contains $\varphi(G)^0$. Maximality of the connected component implies that it is equal to $\varphi(G)^0$. □

Proposition 6.0.12. *Let $\{X_i\}$ be a family of irreducible varieties together with maps $\phi_i : X_i \to G$. Let H be the minimal closed subgroup of G containing all the images $Y_i = \phi_i(X_i)$. Assume that all Y_i contain the identity element. Then:*

1. *H is connected.*
2. *We can choose finitely many among the Y_i, say $Y_{i_1}, \dots, Y_{i_n}$ and signs $\epsilon(i)$ such that $H = Y_{i_1}^{\epsilon(1)} \cdots Y_{i_n}^{\epsilon(n)}$.*

Proof. We may assume that each Y_i^{-1} occurs among the Y_j. Given a multi-index $a = (a(1), \dots, a(n))$ let

$$Y_a = Y_{a(1)} Y_{a(2)} \cdots Y_{a(n)}.$$

We note that each Y_a is constructible and irreducible, being the image of X_a and so are $\overline{Y_a}$. We also have $Y_a Y_b = Y_{(a,b)}$.

Since $Y_b Y_c \subset Y_{(b,c)}$ also $Y_b Y_c \subset \overline{Y_{(b,c)}}$. Let $u \in Y_a$ then $uY_c \subset \overline{Y_{(b,c)}}$ and so $\overline{uY_c} = u \cdot \overline{Y_c} \subset \overline{Y_{(b,c)}}$. It follows that $Y_b \overline{Y_c} \subset \overline{Y_{(b,c)}}$. Repeating the argument, we find

$$\overline{Y_b} \cdot \overline{Y_c} \subset \overline{Y_{(b,c)}}.$$

Let us take Y_a of maximal dimension. Then $\overline{Y_a} \subset \overline{Y_a Y_b Y_{(a,b)}}$. Maximality gives that $\overline{Y_a} = \overline{Y_{(a,b)}}$ and $\overline{Y_b} \subset \overline{Y_a}$. Applying this to $b = a$ we conclude that Y_a is closed under multiplication and applying it to b such that $Y_b = Y_a^{-1}$ we conclude that it is closed under inverse too and contains every Y_b. Thus, $\overline{Y_a}$ is a closed subgroup and has properties (i) and (ii). Since Y_a is constructible, it contains a sent open and dense in $\overline{Y_a}$ and so, by Lemma 6.0.9, $\overline{Y_a} = Y_a Y_a = Y_{(a,a)}$ and so $H = Y_{(a,a)}$ has all the properties stated. □

Since a closed connected subgroup is irreducible, the following corollary holds.

Corollary 6.0.13. *If the X_i are a family of closed connected subgroups of G then the subgroup H generated by them is closed and connected. There's a choice of $G_{i_1}, \dots, G_{i_n}$ among them (perhaps with repetitions!) such that $H = G_{i_1} \cdots G_{i_n}$.*

Exercise 5. *If H and K are closed subgroups of G, one of which is connected then (H, K) - the subgroup of G generated by all the commutators $xyx^{-1}y^{-1}, x \in H, y \in K$ - is closed and connected.*

Exercise 6. *Prove that the symplectic group is connected. You may want to use transvections for that.*

7. Group actions and linearity versus affine

Recall that **a group G acts on a variety V** if we are given a morphism

$$a : G \times V \to V, \quad a(g, v) =: gv,$$

such that $1v = v, \forall v \in V$ and $g(hv) = (gh)v$, $\forall g, h \in G, v \in V$.

The **orbit** of $v \in V$ is Gv and the **isotropy group**, or **stabilizer**, of v is

$$G_v = \{g \in G : gv = v\}.$$

It is a closed subgroup of G. If there is v such that $Gv = V$ we say that G acts **transitively** and that V is a **homogeneous space under** G.

Example 7.0.14. (1) The following maps define group actions

$$G \times G \to G, \qquad a(x, y) = xy.$$

This group action is transitive: there is one orbit. In fact, it is simply transitive: the isotropy groups are trivial. Thus, this is an example of a **principle homogenous space** for G.

Another action is:

$$G \times G \to G, \qquad a(x, y) = xyx^{-1}.$$

Here the orbits Gy are conjugacy classes and the isotropy group of y (or its stabilizer) is its centralizer.

(2) Let V be a vector space over k. A homomorphism

$$r : G \to \mathrm{GL}(V)$$

of algebraic groups over k is called a **rational representation over** k. It defines an action

$$G \times V \to V, \qquad a(g, v) = r(g)(v).$$

(3) The group GL_n acts on the matrices M_n by

$$\mathrm{GL}_n \times M_n \to M_n, \qquad (X, Y) = XYX^{-1}.$$

Assume that k is algebraically closed then the orbits corresponds to matrices in standard Jordan form.

(4) The natural action of GL_n on an n-dimensional vector space V induces an action

$$\mathrm{GL}_n \times \mathbb{P}(V) \to \mathbb{P}(V),$$

where $\mathbb{P}(V)$ is the projective space associated to V. If $V = k^n$ then

$$\mathbb{P}(V) = \{(x_1 : \dots, x_n) : x_i \in k, \exists i\ x_i \neq 0\},$$

where, as usual $(x_1 : \dots, x_n) = (y_1 : \dots, y_n)$ if and only if there's a $\lambda \in k^\times$ such that $\lambda x_i = y_i, \forall i$. We can also say that $\mathbb{P}(V)$ is the space of orbits for the action

$$\mathbb{G}_m \times V \to V, \qquad (\lambda, (x_1, \dots, x_n)) \mapsto (\lambda x_1, \dots, \lambda x_n).$$

This is a special case of the action of GL_n on flag spaces we have already discussed.

Proposition 7.0.15. *Let G act on V.*

(1) *An orbit Gv is open in its closure. It is thus a locally closed set of V.*
(2) *There exist closed orbits.*

Proof. Gv is a constructible set, hence there is a set U such that $U \subset Gv \subset \overline{Gv}$ and U is open in $\overline{Gv}$. But then $Gv = \cup_{g \in G} gU$ is also open.

It follows that $Bd_v := \overline{Gv} - Gv$ is closed and is a union of orbits of G. That is, if $x \in \overline{Gv}$ then $gx \in \overline{Gv}$. This is true because $gx \in g \cdot \overline{Gv} = \overline{gGv} = \overline{Gv}$. In a quasi-projective variety every family of closed sets contains a minimal one. Thus, there is a minimal element in the family $\{Bd_v : v \in V\}$. Take that minimal element $v(0)$. If $Bd_{v(0)}$ is not empty, it is a union of orbits and so there is $v(1)$ such that $G \cdot v(1) \subset Bd_{v(0)}$, but then so is $\overline{G \cdot v(1)}$ and so $Bd_{v(1)} \subsetneqq Bd_{v(0)}$ (because $v(1) \notin Bd_{v(1)}$). That is a contradiction. Thus, $Bd_{v(0)}$ is empty, which means that $G \cdot v(0)$ is closed.

□

Assume from now on that G is an affine algebraic group over k. We shall ultimately prove that G is linear, namely that G is isomorphic to a closed subgroup of GL_n over k. The converse is clear. In order to do so, we shall analyze the action of G on spaces of functions on V, and ultimately take $V = G$ itself.

7.1. Assume that G and V are affine. Then, to give a morphism $G \times V \to V$ corresponds to giving a k-algebra map

$$a^* : k[V] \to k[G] \otimes_k k[V = k[G \times V].$$

The map a^* has additional properties expressing the axioms of the group action, but, at any rate, being the pull-back map on functions, we have

$$a^*(f)(g, v) = f(gv).$$

Define now an action of G on functions:

$$s(g)(f)(x) = f(g^{-1}x).$$

This gives a linear representation

$$G \to \mathrm{GL}(k[V]).$$

Proposition 7.1.1. *Let U be a finite dimensional subspace of $k[V]$.*

(1) *There is a finite dimensional subspace W of $k[V]$ such that W contains U and W is invariant under the action of G via s.*
(2) *U is stable under G if and only if $a^*U \subseteq k[G] \otimes_k U$.*

Proof. It is enough to prove it when U is one dimensional, say $U = kf$. Suppose that

$$a^*f = \sum r_i \otimes f_i \in k[G] \otimes k[V].$$

Then, $s(g)f(v) = a^*(f)(g^{-1}, v) = \sum r_i(g^{-1}) \cdot f_i(v)$ and so the function $s(g)f$ is equal to $\sum r_i(g^{-1}) \cdot f_i$ and we see that $s(g)f \in \mathrm{Span}(\{f_i\}_i)$. The subspace W spanned by all the functions $s(g)f$ is invariant under the action of G and on the other hand, is contained in $k[\{f_i\}]$, which is finite dimensional.

Let now U be a subspace such that $a^*(U) \subset k[G] \otimes U$. Then, if $f \in U$ we have $a^*f = \sum r_i \otimes f_i \in k[G] \otimes U$ and so $s(g)f = \sum r_i(g^{-1})f_i \in U$ and so U is invariant under G.

Conversely, suppose that U is invariant under G and let $f \in U$. Let $\{f_i\}$ be a basis of U and complete it to a basis $\{f_i\} \cup \{g_j\}$ to $k[V]$. For $f \in U$ we may write

$$a^*f = \sum r_i \otimes f_i + \sum t_j \otimes g_j.$$

Then $s(g)f = a^*f(g^{-1}, \cdot) = \sum r_i(g^{-1})f_i + \sum t_j(g^{-1})g_j$ is an element of U and it follows that $t_j(g^{-1}) = 0$ for all $g \in G$ and so that $t_j = 0$. Thus, $a^*f \in k[G] \otimes U$. $\square$

Theorem 7.1.2. *Let G be an affine algebraic group then G is linear, namely G is isomorphic to a closed subgroup of GL_n for some n.*

Proof. Here we choose to work with the action of G given by

$$\rho(g)(f)(x) = f(xg),$$

instead of the action considered above of $s(g)(f)(x) = f(g^{-1}x)$. This is just for notational convenience. It is clear that the same results we have proved above also hold for this action.

G is finitely presented k-algebra. Thus,

$$k[G] = k[f_1, \ldots, f_n],$$

for some f_i. The vector space spanned by the f_i is contained in a G-stable vector space, and so we may assume that $\mathrm{Span}_k\langle f_1, \ldots, f_n\rangle$ is G-invariant. Thus,

$$\rho(g)f_i = \sum_{j=1}^{n} m_{ji}(g)f_j, \qquad \forall g \in G, \tag{7.1.1}$$

where m_{ji} are in $k[G]$. The map

$$g \mapsto \phi(g) = (m_{ij}(g))_{1 \le i,j \le n}$$

is a linear representation of G into GL_n - namely, it is an algebraic homomorphism. We would like to prove that

$$G \cong \phi(G).$$

For that it is enough to show that ϕ is injective, the image is closed and that the coordinate ring of the image is isomorphic to $k[G]$.

Note that since $G \to \mathrm{GL}_n$ is a homomorphism of algebraic groups, the image $\phi(G)$ is a closed subgroup of G. The map ϕ^* simply takes the coordinate T_{ij} of a matrix and sends it to m_{ij}. Since, by (7.1.1),

$$f_i(\cdot) = \sum_j m_{ji}(\cdot) f_j(e),$$

the map ϕ^* is surjective. This implies that ϕ is injective, because given $g \neq g'$ in G, there is some function f of $k[G]$ such that $f(g) \neq f(g')$. If this f is of the form $\phi^* F$ then $F(\phi(g)) \neq F(\phi(g'))$ and so $g \neq g'$.

Now, by definition, $\mathrm{Ker}(\phi^*)$ is the ideal whose radical defines the coordinate ring of the closure of $\phi(G)$ (which is just $\phi(G)$). Since we know $k[G] = k[\mathrm{GL}_n]/\mathrm{Ker}(\phi^*)$ it follows that $\mathrm{Ker}(\phi^*)$ is a radical ideal. Thus, the coordinate ring of $k[\phi(G)]$ is $k[\mathrm{GL}_n]/\mathrm{Ker}(\phi^*) = k[G]$. That means that ϕ is an isomorphism onto its image. $\square$

8. Jordan Decomposition

8.1. Recall: linear algebra. Let k be an algebraically closed field and V a finite dimensional vector space over k. An operator $a \in \mathrm{End}(V)$ is called:

- **Semi-simple**, or **diagonalizable**, if there is a basis of V consisting of eigenvectors.
- **Nilpotent**, if $a^N = 0$ for some $N > 0$.
- **Unipotent**, if $(a-1)^N = 0$ for some $N > 0$.

The Jordan canonical form of a matrix implies that we can write any a as

$$a = a_s + a_n,$$

where a_s is semi-simple, a_n is nilpotent and $a_s a_n = a_n a_s$. The key point of this section is that such a factorization is highly stable under all kind of maps.

Proposition 8.1.1 (Jordan decomposition in $\mathrm{End}(V)$)**.** *Let a be an operator on a finite dimensional vector space V over an algebraically closed field k. Then, we can write*

$$a = a_s + a_n,$$

where a_s is semi-simple, a_n is nilpotent and $a_s a_n = a_n a_s$. Furthermore, such a decomposition is unique and a commutes with a_s, a_n.

Proof. Let $\chi(t)$ be the characteristic polynomial of a. We may write

$$\chi(t) = \prod_i (t - \alpha_i)^{n_i},$$

where the α_i are the distinct eigenvalues of a. This decomposition induces a decomposition of V:

$$V = \oplus V_i, \qquad V_i = \{v \in V : (a - \alpha_i)^{n_i} = 0\}.$$

Using the Chinese Remainder Theorem, we may choose a polynomial $P(t) \in k[t]$ such that

$$P(t) \equiv 0 \pmod{t}, \qquad P(t) \equiv \alpha_i \pmod{(t-\alpha_i)^{n_i}}, \forall i.$$

(Note that this works also if some $\alpha_i = 0$.) Let $Q(t) = t - P(t)$. Consider the linear operators $P(a), Q(a)$. We have

$$a = P(a) + Q(a).$$

$P(a)$ acts as a scalar on each V_i and so $P(a)$ is semi-simple. $Q(a)$ acts as $a - \alpha_i$ on each V_i and so is nilpotent. Furthermore, a commutes with $P(a)$ and $Q(a)$.

Suppose we had another decomposition: $a = b_s + b_n$. Then, since b_s and b_n commute, they also commute with a and so with $P(a), Q(a)$. That is, b_s commutes with a_s and b_n commutes with a_n. The following is easy to prove by linear algebra:

- The sum of two commuting semi-simple operators is semi-simple.
- The sum of two commuting nilpotent operators is nilpotent.

Thus, $a_s - b_s = b_n - a_n$ is both semi-simple and nilpotent, hence must be zero. $\square$

Corollary 8.1.2. *Let $W \subseteq V$ be an a-invariant subspace. Then W is also invariant under a_s and a_n, and so is V/W. Furthermore,*

$$a|_W = a_s|_W + a_n|_W,$$

is the decomposition of $a|_W$ as a sum of a semi-simple operator and a nilpotent operator. The same holds for V/W.

Proof. We have seen in the proof above that $a_s = P(a), a_n = Q(a)$ and that shows that W is stable under a_s, a_n. Note that we can still take P as the polynomial for producing $(a|_W)_s$ and similarly for Q and it follows that $(a|_W)_s = a_s|_W$ and similarly for the nilpotent part. Similar arguments apply to V/W. $\square$

Corollary 8.1.3. *Let a be an operator on V and b an operator on another finite dimensional vector space W. Let $\phi : V \to W$ be a linear map such that*

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "a"'] & W \arrow[d, "b"] \\ V \arrow[r, "\phi"] & W \end{tikzcd}$$

is a commutative diagram. Then also the following diagrams are commutative:

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "a_s"'] & W \arrow[d, "b_s"] \\ V \arrow[r, "\phi"] & W, \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "a_n"'] & W \arrow[d, "b_n"] \\ V \arrow[r, "\phi"] & W. \end{tikzcd}$$

Proof. We can decompose ϕ as

$$V \twoheadrightarrow V/\ker(\phi) \hookrightarrow W,$$

and use the previous corollary. $\square$

Corollary 8.1.4 (Jordan decomposition in $\mathrm{GL}(V)$)**.** *Let $a \in \mathrm{GL}(V)$ then there is a unique decomposition:*

$$a = a_s a_u,$$

such that a_s is semi-simple, a_u is unipotent and $a_s a_u = a_u a_s$. This decomposition if functorial in the sense described in the previous corollaries.

Proof. We have $a = a_s + a_n$. Since a is invertible also a_s must be invertible. This follows from the proof constructing a_s (it acts by a scalar α_i on each V_i and, a being invertible, $\alpha_i \neq 0$ for all i). And so, $a = a_s(1 + a_s^{-1}a_n)$. Since a_s commutes with a_n, the operator $1 + a_s^{-1}a_n$ is unipotent.

Conversely, given a factorization $a = a_s a_u = a_s(1 + (a_u - 1))$ it follows that $a_u - 1$ is nilpotent and so is $a_s(a_u - 1)$ and we get a decomposition $a = a_s + a_s(a_u - 1)$ into a semi-simple and nilpotent operator that commute. It is easy now to deduce uniqueness and functoriality. □

We now want to generalize the Jordan decomposition to infinite-dimensional vector spaces under a finiteness assumption. Let V be a vector space over k and let $a \in \mathrm{End}(V)$. We call a **locally-finite** if V is a union of finite dimensional vector spaces stable under a. a is then called **semi-simple** (**locally nilpotent**) if its restriction of each such subspace is semi-simple (nilpotent). To avoid un-necessary complications, we shall assume that V has countable dimension, or, equivalently, that there are a-invariant finite-dimensional subspaces

$$V_0 \subset V_1 \subset V_2 \subset \dots, \qquad V = \cup_i V_i.$$

In this case, we can define a_s and a_n as the unique operators whose restrictions to each V_i is the semi-simple and nilpotent part, respectively, of a restricted to V_i. We have then,

$$a = a_s + a_n, \qquad a_s a_n = a_n a_s.$$

Furthermore, a_s, a_n are unique with such properties and preserve each V_i. If a is invertible, then we have a factorization

$$a = a_s a_u, a_s a_u = a_u a_s,$$

where a_s is semi-simple and a_u is unipotent, and, again, this factorization is unique.

8.2. Jordan decomposition in linear algebraic groups. Let G be a linear algebraic group. The representation

$$\rho : G \to \mathrm{GL}(k[G]), \qquad (\rho(g)f)(x) = f(xg),$$

maps elements of G to locally finite endomorphisms. Thus, for every $g \in G$ we have a decomposition:

$$\rho(g) = \rho(g)_s \rho(g)_u.$$

The idea is to use this to define the semisimple and unipotent part of g itself and moreover prove that this decomposition is canonical.

Theorem 8.2.1. (1) *There are unique elements g_s, g_u in G such that g_s is semisimple, g_u is unipotent, $g = g_s g_u = g_u g_s$ and. moreover,*

$$\rho(g)_s = \rho(g_s), \quad \rho(g)_u = \rho(g_u).$$

(2) *If $\phi : G \to H$ is a homomorphism of algebraic groups then $\phi(g_s) = \phi(g)_s$ and $\phi(g_u) = \phi(g)_u$.*

(3) *For $G = \mathrm{GL}_n$ the decomposition defined here agrees with the Jordan decomposition defined before.*

Proof. Multiplication on $k[G]$ is a k-algebra homomorphism

$$m : k[G] \otimes_k k[G] \to k[G].$$

Since $\rho(g)$ is a k-algebra homomorphism of $k[G]$ it commutes with multiplication in the sense that

$$m \circ (\rho(g) \otimes \rho(g)) = \rho(g) \circ m.$$

If we apply the functoriality of Jordan decomposition to the vector spaces $k[G] \otimes_k k[G]$ and $k[G]$, relative to the map m, we find that

$$m \circ (\rho(g)_s \otimes \rho(g)_s) = \rho(g)_s \circ m, \qquad m \circ (\rho(g)_u \otimes \rho(g)_u) = \rho(g)_u \circ m. \tag{8.2.1}$$

We are using here the facts, left as exercises that for two linear maps a, b we have $a_s \otimes b_s$ is semi-simple $a_u \otimes b_u$ is unipotent, which implies $(a \otimes b)_s = a_s \otimes b_s$ and $(a \otimes b)_u = a_u \otimes b_u$.

From now on we just prove the statements for the semi-simple part; the proof for the unipotent part is the same. The identity (8.2.1) means that $\rho(g)_s$ is not just a linear map;

it commutes with multiplication. That is, $\rho(g)_s$ is a k-algebra homomorphism of $k[G]$. We can thereferore define a k-algebra homomorphism

$$k[G] \to k, \qquad f \mapsto (\rho(g)_s f)(e). \tag{8.2.2}$$

Such a homomorphism is nothing else but a k-point of G, which we call g_s. We have an equality of homomorphisms

$$f \mapsto (\rho(g)_s f)(e) \quad \text{is the homomorphism} \quad f \mapsto f(g_s).$$

But, on the other hand,

$$f(g_s) = (\rho(g_s)f)(e).$$

Thus, for any function f, we have

$$(\rho(g)_s f)(e) = (\rho(g_s)f)(e).$$

Given a function f we can apply this relation to the functions $\lambda(h)f(x) := f(h^{-1}x)$, $h \in G$, to get:

$$\rho(g)_s(\lambda(h)f)(e) = \rho(g_s)(\lambda(h)f)(e).$$

Now, since $\rho(g) \circ \lambda(h) = \lambda(h) \circ \rho(g)$, if we think of $\lambda(h)$ as a k-linear map on the vector space $k[G]$, we may conclude by the functoriality of Jordan decomposition that

$$\rho(g)_s \circ \lambda(h) = \lambda(h) \circ \rho(g)_s.$$

Thus, we have the relation

$$\lambda(h)(\rho(g)_s f)(e) = \rho(g)_s(\lambda(h)f)(e) = \rho(g_s)(\lambda(h)f)(e).$$

The left hand side is $\rho(g)_s f(h^{-1})$, while the right hand side is $f(h^{-1}g_s) = (\rho(g_s)f)(h^{-1})$. Thus,

$$\rho(g)_s f = \rho(g_s)f, \qquad \forall f \in k[G].$$

This means that $\rho(g)_s = \rho(g_s)$. The same holds for unipotent parts: $\rho(g)_u = \rho(g_u)$. Since ρ is faithful, that finishes the proof of (1).

We now prove (2). We may factor ϕ as

$$G \twoheadrightarrow \phi(G) \hookrightarrow H.$$

(Note that $\phi(G)$ is a closed subgroup of H and so all the groups here are affine.) In the first case, ϕ^* identifies $k[\phi(G)]$ with a subspace T of $k[G]$. The operator $\rho(g)|_T$ is nothing else then $\rho(\phi(g))$ (via ϕ^*). Functoriality of Jordan decomposition for restriction gives $\rho(g)_s = \rho(g_s)$ is equal to $\rho(\phi(g))_s = \rho(\phi(g)_s)$ on T and so, since ρ is faithful, $\phi(g_s) = \phi(g)_s$.

For $\phi(G) \hookrightarrow H$ we may view $k[G]$ as $k[H]/I$ where I is the ideal defining the closed subset $\phi(G)$. The ideal I is stable under the action of $\rho(\phi(G))$. Now the result will follow from functoriality of Jordan decompositions for quotient spaces.

It remains to prove (3). G acts naturally on the vector space $V = k^n$. Fix a non-zero function $f \in V^*$ - the dual vector space. For every $v \in V$ define a function f_v on G by

$$f_v(g) = f(gv).$$

This gives us an injective linear map

$$V \to k[G], \qquad v \mapsto f_v.$$

Now, f_{gv} is the function whose value at h is $f(hgv) = (\rho(g)f_v)(h)$. Thus, the map $V \to k[G]$ is equivariant for the natural action of G on V and on $k[G]$ via ρ. Functoriality now gives us $\rho(g)_s = \rho(g_s)$. □

Corollary 8.2.2. *An element $g \in G$ is semi-simple (resp. unipotent) if and only if for any isomorphism ϕ from G to a closed subgroup of some GL_n, $\phi(g)$ is semi-simple (resp. unipotent).*

Proposition 8.2.3. *The set of unipotent elements of G is a closed subset.*

Proof. Choose an embedding $G \to \mathrm{GL}_n$. The image is closed and the map takes unipotent to unipotent. Thus, it is enough to prove the statement for $G = GL_n$. But then the unipotent elements are defined by the equation $(a-1)^n = 0$. □

Theorem 8.2.4. *Let G be a subgroup of GL_n consisting of unipotent elements. Then G can be conjugated to U_n, where U_n are the upper unipotent matrices. That is, for some $x \in G_n$ we have*

$$xGx^{-1} \subset U_n.$$

Proof. One argues by induction on n. The case $n = 1$ being clear. Assume $n > 1$. If $V = k^n$ is a reducible representation of G then there is a proper G-stable subspace W. The action of G on W and V/W is unipotent and so there are bases B for W and C' to V/W relative to which the action is by upper unipotent matrices. Lift the elements of C' in any way to V obtaining a set C. Then $B \cup C$ is a basis for V in which the action of G is by upper unipotent matrices.

If V is an irreducible representation of G then, by a theorem of Burnside (that is not at all obvious; it follows from Jacobson's density theorem. See Lang/Algebra, Chapter XVII) the linear span of G is the whole of $\mathrm{End}(V)$.

Consider the linear functional $g \mapsto \mathrm{Tr}(g)$. Since $g - 1$ is nilpotent, $\mathrm{Tr}(g-1) = 0$ and so this functional is constant on G, having value $n = \dim(V)$. Let $x, y \in G$ and write $x = 1 + N$, where N is nilpotent. We have $\mathrm{Tr}(y) = \mathrm{Tr}(xy)$, because $x, xy \in G$ and $\mathrm{Tr}(xy) = \mathrm{Tr}((1+N)y) = \mathrm{Tr}(y) + \mathrm{Tr}(ny)$. Thus, $\mathrm{Tr}(Ny) = 0$ for all $y \in G$. But then this holds for all y in the k-linear span of G, which is $M_n(k)$. If we choose y to vary over the elementary matrices (zero everywhere except for 1 in a single place) and calculate $\mathrm{Tr}(Ny)$ we find that $N = 0$ and so $G = \{1\}$. This is a contradiction to irreducibility. $\square$

The group U_n is nilpotent. To see that let $U(a)$ be the matrices M with zero on all the diagonals $m_{i+x,j+x}$ with $0 \leq j - i \leq a - 1$. One checks that $U(a)U(b) \subseteq U(a+b)$ and that $1 + U(a)$ is a group. One the proves that if $G^{(0)} = U_n, G^{(1)} = [U_n, U_n], \dots G^{(\ell+1)} = [U_n, G^{(\ell)}]$ then $G^{(\ell)} \subseteq 1 + U(\ell + 1)$. One proves by induction, using that if u is nilpotent then $(1+u)^{-1} = 1 - u + u^2 - u^3 + \dots$ (this is a finite sum!), and by expanding the expression

$$[1+u, 1+v] = (1+u)(1+v)(1-u+u^2-\dots)(1-v+v^2-\dots).$$

Corollary 8.2.5. *A unipotent algebraic group is nilpotent, hence solvable (both notions in the sense of group theory).*

Proof. We can embed the group into GL_n for some n and so into U_n. Since U_n is nilpotent so is any subgroup of it. A nilpotent group is solvable. $\square$

Corollary 8.2.6. *Let G be a unipotent algebraic group and $\rho : G \to \mathrm{GL}_n$ a rational representation. Then there is a non-zero fixed vector for this action.*

Proof. This follows directly from the theorem. $\square$

Corollary 8.2.7. *Let G be a unipotent algebraic group and V an affine variety on which G acts. Then all the orbits of G in V are closed.*

Proof. Let W be a G-orbit. We may assume without loss of generality that W is dense in V. We want to show that $W = V$. We know that W is open in V. Let $C = V - W$ and let I the ideal defining the closed set C. To show C is empty it is enough to show that I contains a non-zero constant function. The group G acts on I and the representation is locally finite as $I \subset k[V]$. Thus, there is a non-zero fixed function f in I. This means that f is constant on W, hence on V, hence a constant. $\square$

9. Commutative algebraic groups

9.1. Basic decomposition.

Lemma 9.1.1. *Let V be a finite dimensional vector space over an algebraically closed field k. Let $S \subset \mathrm{End}(V)$ be a set of commuting operators, then there is a basis of V in which all the semi-simple elements of S are diagonal and all the elements of S are upper-triangular.*

Proposition 9.1.2. *Let G be a commutative linear algebraic group over an algebraically closed field k. The set of semi-simple elements of G, G_s, and the set of unipotent elements of G, G_u, are both closed subgroups of G. Further,*

$$G \cong G_s \times G_u.$$

Proof. We may assume that G is a closed subgroup of $\mathrm{GL}(V)$ for some vector space V over k of dimension $n < \infty$. A collection of commuting linear maps can be simultaneously triangularized. Thus, we may assume that $G \subseteq B_n$, where B_n is the standard Borel subgroup of GL_n. It is easy to see that an element of B_n is semi-simple iff it belongs to the torus T_n, and unipotent iff it belongs to the unipotent group U_n. Thus, $G_s = G \cap T_n$, $G_u = G \cap U_n$ and thus both are closed subgroups.

The existence of Jordan decomposition shows that the map $G_s \times G_u \to G, (g_s, g_u) \mapsto g_s g_u$ is surjective. Since $T_n \cap U_n = \{I_n\}$ the map is also injective. The set-theoretic maps $G \to G_s, G \to G_u$ taking an element g to g_s and g_u, respectively, are morphisms because they are defined by taking a subset of the coordinates of g. Thus, we have an isomorphism $G_s \times G_u \cong G$. □

The proposition reduces the study of commutative linear algebraic groups to the study of semisimple commutative groups and unipotent commutative groups.

9.2. Diagonalizable groups and tori. For an algebraic group G, the group of **characters** of G is

$$X^*(G) = \mathrm{Hom}_k(G, \mathbb{G}_m) \qquad \text{(homomorphism of } k\text{-algebraic groups).}$$

It is a commutative (abstract group) which we write additively:

$$(\chi_1 + \chi_2)(g) := \chi_1(g) \cdot \chi_2(g).$$

The **cocharacters** of G, also called **one-parameters subgroups** of G, are

$$X_*(G) := \mathrm{Hom}_k(\mathbb{G}_m, G) \qquad \text{(homomorphism of } k\text{-algebraic groups).}$$

In general this is only a set. If G is a commutative group then $X_*(G)$ is a commutative group as well, and again the group is written additively. We use the notation $n\chi$ for $n \in \mathbb{Z}$ to denote the cocharacter

$$(n\chi)(x) = \chi(x)^n = \chi(x^n).$$

Proposition 9.2.1. *Let $x \in T_n$ and write $x = (\chi_1(x), \dots, \chi_n(x))$. Then each χ_i is a character of T_n. We have*

$$k[T_n] = k[\chi_1^{\pm 1}, \dots, \chi_n^{\pm 1}].$$

The functions $\chi_1^{a_1} \cdots \chi_n^{a_n}$ are linearly independent over k and are all characters of T_n. Furthermore, every character of T_n is of this form. We have $X^*(T_n) \cong \mathbb{Z}^n$ and $X_*(T_n) \cong \mathbb{Z}^n$.

Proof. The statement $k[T_n] = k[\chi_1^{\pm 1}, \dots, \chi_n^{\pm 1}]$ is equivalent to the statement in case $n = 1$ (which is clear), together with the statement $T_n \cong \mathbb{G}_m^n$, which is also clear.

We have analyzed before $\mathrm{End}(T_n)$ and the arguments made there easily extend to $\mathrm{Hom}(\mathbb{G}_m^a, \mathbb{G}_m^b)$ and so give us the statements about the characters. And about the cocharaters. $\square$

We say that a linear algebraic group is **diagonalizable** if it is isomorphic to a closed subgroup of T_n for some n.

Theorem 9.2.2. *The following are equivalent:*

(0) *G is commutative and $G = G_s$.*
(1) *G is diagonalizable.*
(2) *$X^*(G)$ is a finitely generated abelian group and its elements form a k-basis for $k[G]$.*
(3) *Any rational representation of G is a direct sum of one dimensional representations.*

Proof. The equivalence of (0) and (1) is clear from Lemma 9.1.1. Assume that G is diagonalizable, say $G \subseteq T_n$ a closed subgroup. Thus, $k[G] = k[T_n]/I$ for some ideal I. By Dedekind's theorem on independence of characters, the elements of $X^*(G)$ are linearly independent over k and they contain the spanning set obtained as the restriction of the characters from $X^*(T_n)$ (since $k[T_n]$ has a basis consisting of the characters $\chi_1^{a_1}\cdots\chi_n^{a_n}$, and the restriction of a character is a character). Thus, $X^*(G)$ form a basis for $k[G]$ and

$$X^*(T_n)\to X^*(G),$$

is surjective and, in particular, $X^*(G)$ is finitely generated.

Let us assume now that G has the property that $X^*(G)$ is a finitely generated abelian group and its elements form a k-basis for $k[G]$. Let ϕ be a rational representation of G, $\phi : G\to \mathrm{GL}_n$. Consider the function

$$g\mapsto \phi(g)_{ij}.$$

It is a regular function on G, and so there are scalars $m(\chi)_{ij}$, almost all of which are zero, such that

$$\phi(g)_{ij}=\sum_\chi m(\chi)_{ij}\cdot\chi(g).$$

Packing this together, we find matrices $M(\chi)$, almost all of which are zero, such that

$$\phi(g)=\sum_\chi \chi(g)M(\chi).$$

We then have

$$\phi(gh)=\sum_\chi \chi(gh)M(\chi)=\sum_\chi \chi(g)\chi(h)M(\chi).$$

On the other hand,

$$\phi(g)\phi(h)=(\sum_\chi \chi(g)M(\chi))(\sum_\chi \chi(h)M(\chi))=\sum_{\chi_1,\chi_2}\chi_1(g)\chi_2(h)M(\chi_1)M(\chi_2).$$

We can view the two formulas for $\phi(gh)=\phi(g)\phi(h)$ as formulas between characters on $G\times G$ (viz., we have characters $(g,h)\mapsto\chi(g)\chi(h)$ and $(g,h)\mapsto\chi_1(g)\chi_2(h)$). Using independence of characters, we find that

$$M(\chi_1)M(\chi_2)=\delta_{\chi_1,\chi_2}M(\chi_1).$$

In addition, $1=\phi(1)=\sum_\chi M(\chi)$ and so we have decomposed the identity operator into a sum of (commuting) orthogonal idempotents $\{M(\chi)\}$. Let

$$V(\chi)=\mathrm{Im}(M(\chi)).$$

One easily check that

$$V=\oplus_\chi V(\chi),$$

and that G acts on each $V(\chi)$ via the character χ. This proves (3).

Assume (3) and choose an embedding $G \subseteq \mathrm{GL}_n$, realizing G as a closed subgroup. (3) implies that the natural representation of G on k^n affords a basis in which G is diagonal, namely, we can conjugate G in GL_n into T_n and so G is diagonalizable. $\square$

Corollary 9.2.3. *If G is diagonalizable then $X^*(G)$ is a finitely generated abelian group with no p-torsion if $char(k) = p > 0$. The algebra $k[G]$ is isomrophic to the group algebra of $X^*(G)$ and its comultiplication, and coinverse, are given by*

$$\chi \mapsto \chi \otimes \chi, \qquad \chi \mapsto -\chi.$$

This construction can be reversed. Given a finitely generated abelian group M, with no p-torsion if $char(k) = p > 0$, we can define a k-algebra - the group algebra of M - $k[M]$. In order not confuse the operations, we write M additively, but we write $e(m)$ for the corresponding function of $k[M]$ so that $k[M]$ has a basis $\{e(m) : m \in M\}$ and

$$e(m_1)e(m_2) = e(m_1 + m_2).$$

We endow $k[M]$ with comultiplication, coinverse and counit by, respectively,

$$\Delta(e(m)) = e(m) \otimes e(m), \quad \iota(e(m)) = e(-m), \quad e(m) \mapsto 1, \forall m \in M.$$

This construction satisfies

$$k[M_1 \oplus M_2] = k[M_1] \otimes_k k[M_2].$$

We let $\mathscr{G}(M)$ be the corresponding algebraic group. We have

$$\mathscr{G}(M_1 \oplus M_2) \cong \mathscr{G}(M_1) \times \mathscr{G}(M_2).$$

The group $\mathscr{G}(M)$ is a diagonalizable group. To show that it is enough to consider the case where $M = \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$. It not hard to check that then $\mathscr{G}$ is isomorphic to $\mathbb{G}_m$ and to μ_n, respectively.

There is a canonical isomorphism

$$M \cong X^*(\mathscr{G}(M)).$$

Indeed, each $e(m)$ is a character of $\mathscr{G}(M)$ (as follows from $\Delta(e(m)) = e(m) \otimes e(m)$). Any character of $\mathscr{G}(M)$ is a linear combination of the characters $e(m)$ and so, as before, it must be one of the characters $e(m)$.

If G is a diagonalizable group then

$$G \cong \mathscr{G}(X^*(G)),$$

as we have seen.

From this constructions one deduces the following:

Corollary 9.2.4. *Let G be a diagonalizable group.*

(1) *G is isomorphic to a direct product of a torus and a finite abelian group of order prime to $p = char(k)$.*
(2) *G is connected if and only if it is a torus.*
(3) *G is a torus if and only if $X^*(G)$ is a free abeian group.*

We have constructed a correspondence

$$\{\text{f.g. abelian groups with no } p\text{-torsion}\} \longleftrightarrow \{\text{diagonalizable groups}\}.$$

In fact, this correspondence is an equivalence of categories. Given a morphism of diagonalizable groups $G_1 \to G_2$ we get a homomorphism of groups $X^*(G_2) \to X^*(G_1)$ and conversely. This is left as an exercise. Note that the categories are not abelian; the quotient of an abelian group with no p-torsion by a subgroup may well have p-torsion. That complicates the picture a little bit. Note that this equivalence implies that $\mathrm{Hom}(\mathbb{G}_m^a, \mathbb{G}_m^b) \cong M_{a,b}(\mathbb{Z})$.

9.2.1. *Galois action.* The picture can be made richer taking into account the Galois action. Let F be a field and k its separable closure. An F-group G is called diagonalizable if $G(k)$ is. One can show that $G(k)$ is diagonalizable iff $G(F^{alg})$ is. Let $X^*(G) := X^*(G(k)) = X^*(G(F^{alg}))$. The Galois group $\Gamma = \mathrm{Gal}(k/F)$ acts on $X^*(G)$ as follows. Let $\gamma \in \Gamma, \chi \in X^*(G)$:

$${}^{\gamma}\chi(x) = \gamma(\chi(\gamma^{-1}(x))).$$

Then ${}^{\gamma}\chi$ is a group homomorphism. It is also a morphism: If we choose a presentation $k[G] = k[x_1, \dots, x_n]/(\{f_j\})$ and χ is any function $G \to \mathbb{A}^1$ we can make the same definition for ${}^{\gamma}\chi$. Then, letting $\chi(x_1, \dots, x_n) = \sum_I c_I \cdot x^I$ we have

$${}^{\gamma}\chi(a_1, \dots, a_n) = \gamma(\sum_I c_I \cdot (\gamma^{-1}(a))^I) = \sum_I \gamma(c_I) \cdot a^I.$$

That is,

$$\chi = \sum_I c_I \cdot x^I \;\Rightarrow\; {}^\gamma\chi = \sum_I \gamma(c_I) \cdot x^I.$$

Finally, we clearly have

$${}^\alpha({}^\beta\chi) = {}^{\alpha\beta}\chi.$$

This makes $X^*(G)$ into a continuous Γ-module. The main theorem is that there is an antiequivalence of categories

$$\{\Gamma\text{-modules that are f.g. abelian groups with no } p\text{-torsion}\} \longleftrightarrow \{\text{diagonalizable } F\text{-groups}\}.$$

Example 9.2.5. Let K/L be a finite separable extension of fields. Let $G = \mathrm{Res}_{K/L}\mathbb{G}_m$. Recall that this is the algebraic group over L associating to an L-algebra R the group

$$G(R) = (K \otimes_L R)^\times.$$

Choose an algebraic closure $\bar{K}$ of K. It is also an algebraic closure of L. We have

$$G(\bar{K}) = (K \otimes_L \bar{K})^\times \cong \oplus_{\tau \in \mathrm{Hom}_L(K,\bar{K})} \bar{K}^\times.$$

The isomorphism is given on generators by

$$\alpha \otimes \lambda \mapsto (\tau(\alpha)\lambda)_\tau.$$

This is the isomorphism showing that G is a torus of dimension $[K : L]$.

Let us write χ_τ for the character

$$\chi_\tau(\alpha \otimes \lambda) = \tau(\alpha)\lambda$$

(its values on a general invertible element $\sum \alpha_i \otimes \lambda_i$ is $\sum \tau(\alpha_i)\lambda_i$). To determine the Galois action it is enough to consider generators. Let $\sigma \in \mathrm{Gal}(\bar{L}/L)$ then

$$\begin{aligned}
{}^\sigma\chi_\tau(\alpha \otimes \lambda) &= \sigma(\chi_\tau(\sigma^{-1}(\alpha \otimes \lambda))) \\
&= \sigma(\chi_\tau(\alpha \otimes \sigma^{-1}(\lambda))) \\
&= \sigma(\tau(\alpha) \cdot \sigma^{-1}(\lambda)) \\
&= (\sigma \circ \tau)(\alpha) \cdot \lambda \\
&= \chi_{\sigma\tau}(\alpha \otimes \lambda).
\end{aligned}$$

Therefore,

$${}^\sigma\chi_\tau = \chi_{\sigma\tau}.$$

Since $\mathrm{Gal}(\bar{L}/L)$ acts transitively on $\mathrm{Hom}_L(K, \bar{K})$, and so on the set $\{\chi_\tau : \tau \in \mathrm{Hom}_L(K, \bar{K})\}$, one concludes that as a Galois module

$$X^*(\mathrm{Res}_{K/L}\mathbb{G}_m) = \mathrm{Ind}_{\mathrm{Gal}(\bar{L}/K)}^{\mathrm{Gal}(\bar{L}/L)} 1.$$

(The induction here is in a more subtle sense than in the theory of representations on vector spaces; it is at the level of lattices.)

Here is a particular situation. The **Deligne torus** is defined as

$$\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m.$$

This is a 2-dimensional torus defined over $\mathbb{R}$. Let us write $\mathbb{C} = \mathbb{R}\cdot 1 + \mathbb{R}\cdot i$. Then $x + iy$ is invertible iff $x^2 + y^2 \neq 0$. The group $\mathbb{S}$ was then constructed as the affine variety

$$\mathbb{R}[x, y, (x^2 + y^2)^{-1}].$$

Since, $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i$, the co-multiplication of $\mathbb{S}$ is given by

$$x \mapsto x \otimes x - y \otimes y, \qquad y \mapsto x \otimes y + y \otimes x.$$

Over $\mathbb{C}$ we have the isomorphism

$$\mathbb{S} \cong \mathbb{G}_m^2, \qquad (x, y) \mapsto (x + iy, x - iy).$$

(Note that since $0 \neq x^2 + y^2 = (x + iy)(x - iy)$ the right hand side is indeed in $\mathbb{G}_m^2$.) The morphism is clearly invertible and one verifies directly that it is a homomorphism. A basis for the characters of $\mathbb{S}$ is thus visibly

$$(x, y) \overset{\chi_1}{\mapsto} x + iy, \qquad (x, y) \overset{\chi_\sigma}{\mapsto} x - iy.$$

Here σ is a formal symbol, but now let σ denote complex conjugation. Then

$$\begin{aligned} {}^\sigma\chi_1(x, y) &= \sigma(\chi_1(\sigma^{-1}(x, y))) \\ &= \sigma(\chi_1(\bar{x}, \bar{y})) \\ &= \sigma(\bar{x} + i\bar{y}) \\ &= x - iy \\ &= \chi_\sigma(x, y). \end{aligned}$$

Thus,

$${}^\sigma\chi_1 = \chi_\sigma$$

(justifying our notation) and, necessarily, ${}^{\sigma}\chi_\sigma = \chi_1$. Thus, as a Galois module,

$$X^*(\mathbb{S}) = \mathbb{Z}^2 = \mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_\sigma, \qquad \sigma \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

We have an exact sequence of Galois modules:

$$0 \to \mathbb{Z} \cdot (\chi_1, -\chi_\sigma) \to \mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_\sigma \to \mathbb{Z} \to 0,$$

where $\mathbb{Z}$ on the right is given the trivial Galois action. The map to it is simply $n_1\chi_1 + n_\sigma\chi_\sigma \mapsto n_1 + n_\sigma$. The module $\mathbb{Z}\cdot(\chi_1, -\chi_\sigma)$ is isomorphic to $\mathbb{Z}$ with σ acting as multiplication by -1; it defines a torus T over $\mathbb{R}$.

This exact sequence of Galois module corresponds to an exact sequence of tori over $\mathbb{R}$:

$$1 \to \mathbb{G}_m \to \mathbb{S} \to T \to 1. \tag{9.2.1}$$

Using the $\mathbb{G}_m \times \mathbb{G}_m$ model for $\mathbb{S}(\mathbb{C})$, the complex points of this sequence are

$$1 \to \mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{C}^\times \to 1,$$

where the maps are $z \mapsto (z, z)$ and $(z_1, z_2) \mapsto (z_1/z_2)$. (Check this!). The real points of $\mathbb{S}$ are written in the $\mathbb{G}_m \times \mathbb{G}_m$ model as $\{(z, \bar{z}) : z \in \mathbb{C}^\times\}$. And so the real points of the sequence (9.2.1) are

$$1 \to \mathbb{R}^\times \to \mathbb{C}^\times \to T(\mathbb{R}) \to 1,$$

where the maps are $r \mapsto (r, r)$ and $z \mapsto z/\bar{z} \in \mathbb{C}^1$. One finds that $T(\mathbb{R}) = \mathbb{C}^1$ and the sequence of real points of (**??**) is just

$$1 \to \mathbb{R}^\times \to \mathbb{C}^\times \to \mathbb{C}^1 \to 1.$$

9.3. Action of tori on affine varieties. Let T be a torus. There is a canonical perfect pairing

$$X^*(T) \times X_*(T) \to \mathbb{Z}, \qquad (\chi, \phi) \mapsto \chi \circ \phi \in \mathrm{End}(\mathbb{G}_m) = \mathbb{Z}.$$

Let V be an affine variety on which T acts. Then T has a locally finite rational representation on $k[V]$; letting $k[V]_\chi$ denote the eigenspace corresponding to $\chi \in X^*(T)$, namely the functions f satisfying $s(t)f(v) = f(t^{-1}.v) = \chi(t)f(v)$ for all $t \in T, v \in V$, we have

$$k[V] = \oplus_{\chi \in X^*(T)} k[V]_\chi, \qquad k[V]_\chi k[V]_\psi = k[V]_{\chi+\psi}.$$

Theorem 9.3.1. *Let V be an affine T-variety. For $\lambda \in X_*(T)$, a one-parameter subgroup, put*

$$V(\lambda) = \{v \in V : \textit{the morphism } \lambda : \mathbb{G}_m \to V, g \mapsto \lambda(g).v,$$
$$\textit{extends to a morphism } \lambda' : \mathbb{A}^1 \to V\}.$$

We then say $\lim_{t\to 0} \lambda(t).v$ exists and define it as $\lambda'(0)$. We have that $V(-\lambda)$ is the set of v such that $\lim_{t\to\infty} \lambda(t).v$ exists. Then:

(1) *$V(\lambda)$ is a closed set.*
(2) *$V(\lambda) \cap V(-\lambda)$ is the set of fixed points for* $\mathrm{Im}\lambda$.

Proof. Let $f = \sum_\chi f_\chi$. Then,

$$s(\lambda(a))f = \sum_\chi \chi(\lambda(a)) f_\chi = \sum_\chi a^{\langle \chi,\lambda\rangle} f_\chi.$$

Then $\lim_{a\to 0} \lambda(a).v$ exists iff $\lim_{a\to 0} f(\lambda(a).v)$ exists for every f, iff $\lim_{a\to 0} a^{\langle\chi,-\lambda\rangle} f_\chi(v)$ exists, iff for every χ such that $\langle\chi,-\lambda\rangle < 0$, $f_\chi(v) = 0$. That is, $v \in V(\lambda)$ iff v is a zero of the ideal $\oplus_{\{\chi:\langle\chi,\lambda\rangle>0\}} k[V]_\chi$.

It follows that $V(\lambda) \cap V(-\lambda)$ is the set of v that annihilate all $k[V]_\chi$ with $\langle\chi,\lambda\rangle \neq 0$, which are the set of v such that $f(\lambda(a)v) = f(v)$ for all $a \in k^*$. That is, the fixed points for λ. □

Example 9.3.2. The group $SO_2 = \{\left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right) : a^2 + b^2 = 1\}$ is actually a form of $\mathbb{G}_m$. Let $a = \frac{1}{2}(x + 1/x), b = \frac{-i}{2}(x - 1/x)$. This gives a homomorphism $\mathbb{G}_m \to SO_2$ which is clearly invertible, $x = a + ib$. Given a vector $v = (v_1, v_2)$ we have

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} v_1 \\ v_2\end{pmatrix} = \begin{pmatrix} \frac{1}{2}(x+1/x)v_1 + \frac{-i}{2}(x-1/x)v_2 \\ \frac{i}{2}(x-1/x)v_1 + \frac{1}{2}(x+1/x)v_2\end{pmatrix}.$$

This has a limit as $x \to 0$ iff $v_1 + iv_2 = 0$. It has a limit as $x \to \infty$ iff $v_1 - iv_2 = 0$. And so we find that the only fixed point under SO_2 is the zero vector $(0, 0)$.

Example 9.3.3. Let $\lambda : \mathbb{G}_m \to G$ be a one parameter group and define an action of G on G (playing now the role of the affine variety) by conjugation. So that $\mathbb{G}_m$ acts by $a.g = \lambda(a)g\lambda(a)^{-1}$. In this case the variety $V(\lambda)$ is denoted $P(\lambda)$. One checks that $P(\lambda)$ is a subgroup, hence a closed subgroup. In fact, it is a parabolic subgroup (a notion we discussed for GL_n and discuss in general later). We have $P(\lambda) \cap P(\lambda^{-1})$ equal to the fixed points of λ, which is precisely the centrailizer of $\lambda(\mathbb{G}_m)$ in G.

9.4. Unipotent groups. We shall be very brief here. More information can be found in Springer's book.

Let $k = \bar{k}$ be an algebraically closed field of characteristic $p \geq 0$. A unipotent linear algebraic group G/k is called **elementary** if it is abelian and, moreover, if $p > 0$ then its elements have order dividing p.

Example 9.4.1. The additive group $\mathbb{G}_a = (\mathbb{A}^1, +)$ is isomorphic to the subgroup $\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$ and hence is unipotent. If $p > 0$ then $a + \cdots + a$ (p times) is $pa = 0$. Thus, $\mathbb{G}_a$ is an elementary unipotent group.

Theorem 9.4.2. *Let G be a connected linear algebraic group of dimension 1. Then, either $G \cong \mathbb{G}_m$ or $G \cong \mathbb{G}_a$.*

For the proof see Springer, §3.4. We sketch part of the proof.

- G is commutative. Fix $g \in G$ and consider the morphism $G \to G$, $x \mapsto xgx^{-1}$. The closure of the image is either G, or e, by dimension considerations and connectedness. Assume it is G. Then, every element, but finitely many, is of the form xgx^{-1} and thus (viewing G as a subgroup of some GL_n) and has the same characteristic polynomial as g. Altogether there are finitely many possibilities for the characteristic polynomial of elements of g. Since G is connected, the characteristic polynomial (whose coefficients are algebraic functions on G) must then be constant. However, the identity has characteristic polynomial $(t-1)^n$ and, thus, so does every element of G. Thus, G is unipotent. In that case, it is nilpotent and so its commutator subgroup must be strictly contained in G (and connected), hence equal to $\{e\}$. It follows that G is commutative and that contradicts our assumption that the image of $x \mapsto xgx^{-1}$ is G. Thus, the image is always $\{e\}$ and G is commutative.
- Therefore, $G \cong G_s \times G_u$. Both factors must be connected and exactly one of which 1-dimensional (and so the other must be trivial). If G_s is one dimensional, it is a connected diagonalizable group of dimension one hence isomorphic to $\mathbb{G}_m$. Else, $G \cong G_u$, a commutative unipotent group.
- If $G = G_u$ then G is elementary. One views G as a subgroup of the upper unipotent group U_n in GL_n, for a suitable n. If the characteristic is p, writing $x = 1+N$, where N is nilpotent, using the binomial formula and divisibility of binomial coefficients by p, one concludes that $x^{p^n} = 1$.

Let $G^{(m)}$ the image of the homomorphism $x \mapsto x^m$. It is a closed connected subgroup of G and so is either G or $\{1\}$. If $G^{(p)} = G$ then, inductively, $G^{(p^n)} = (G^{(p^{n-1})})^{(p)} = G$ and that is a contradiction, because $x^{p^n} = 1$ for all $x \in G$. Thus, $G^{(p)} = \{1\}$ and therefore G is elementary.

- The next (hard) step is to show that an elementary unipotent group of dimension 1 is isomorphic to $\mathbb{G}_a$. For that see Springer's book.