EXERCISES FOR THE COURSE MATH 722, ALGEBRAIC GEOMETRY, FALL 2012

Some of the exercises are taken from Hartshorne. It's a good idea to consult the book often and look at all the exercises. It may be the case that to solve an exercise appearing below you would have to solve first an easier exercise appearing in Hartshorne (if you find it useful).

First assignment: Please submit exercises (1) - (6) by Wednesday, September 19, 20;00. Mail solutions in the form

LastName.AG.AssignmentNumber.pdf (so Smith.AG.13.pdf should be the name of the file of Smith's solutions to assignment 13)

to the address 2eyalgoren@gmail.com (and **NOT** to my McGill address).

Second assignment: Please submit exercises (9) - (14) by Wednesday, October 3, 20;00.

Third and Fourth assignment: Please submit exercises (15) - (26), except exercise (22), by Wednesday, October 24, 20;00.

Fifth assignment: Please submit exercises (27) - (31) by Monday, November 5, 20;00.

Sixth assignment: Please submit exercises (32) - (36) by Monday, November 19, 20;00.

(1) Prove the following assertions:
- Let $T \subseteq k[x_1, \dots, x_n]$. Then, $\mathscr{Z}(T) = \mathscr{Z}(\langle T \rangle) = \mathscr{Z}(\sqrt{\langle T \rangle})$.
- For ideals $\mathfrak{a}, \mathfrak{b} \lhd k[x_1, \dots, x_n]$ we have

$$\mathscr{Z}(\mathfrak{a}) \cup \mathscr{Z}(\mathfrak{b}) = \mathscr{Z}(\mathfrak{a}\mathfrak{b}), \qquad \mathscr{Z}(\mathfrak{a}) \cap \mathscr{Z}(\mathfrak{b}) = \mathscr{Z}(\mathfrak{a} + \mathfrak{b}).$$

(2) If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the standard way, prove that the topology on $\mathbb{A}^2$ is not the product topology.

(3) Let Y be the plane curve $y = x^2$ (i.e., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$, the coordinate ring of Y, is isomorphic to the polynomial ring in one variable over k. Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to the polynomial ring in one variable over k. Let W be the plane curve $x^2 + 3y^2 = 1$. Then $A(W)$ is isomorphic to either $A(Y)$ or $A(Z)$. Which is it?

(4) Let $\mathfrak{a}$ be an ideal of $k[x_1, \dots, x_n]$ which can be generated by r elements. Prove that every irreducible component of $\mathscr{Z}(\mathfrak{a})$ has dimension $\geq n - r$.

(5) Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements. Illustrate this for the case $B = k[x^2, xy, y^2]$ (a subring of $k[x, y]$).

(6) Consider the affine space $\mathbb{A}^{n^2}$, where $n \geq 1$ an integer. We think of elements of this space as matrices $(x_{ij})_{i,j=1,\dots,n}$. Let $k \leq n$. Prove that the set of matrices of rank at most k is an algebraic set in $\mathbb{A}^{n^2}$. For $k = n-1$, determine the dimension of these algebraic sets and prove they are irreducible.

(7) Let $V \subset \mathbb{A}^n$ be a variety of codimension 1, that is, a hypersurface. Prove that V is the zero set of an irreducible polynomial.

(8) Prove the following properties of homogonous ideals in a graded ring $R = \oplus_{d=0}^{\infty} R_d$.
   (a) The intersection, sum, product and radical of homogenous ideals are homogenous.
   (b) An ideal $\mathfrak{a}$ is homogenous if and only if $\mathfrak{a}$ is generated by homogenous elements.
   (c) A homogenous ideal $\mathfrak{a}$ is a radical ideal if and only if for each homogenous element f we have $f^n \in \mathfrak{a} \Rightarrow f \in \mathfrak{a}$.
   (d) A homogenous ideal $\mathfrak{a}$ is prime if and only if for any two homogenous elements f, g we have $fg \in \mathfrak{a}$ implies $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.
   (e) If $\mathfrak{a}$ is a homogenous ideal then $R/\mathfrak{a}$ has a natural grading.

(9) Let R be a ring and let $I \lhd R$ be an ideal. Prove that $\oplus_{d=0}^{\infty} I^d$ and $\oplus_{d=0}^{\infty} I^d/I^{d+1}$ are graded rings. Suppose now that $R = k[x_1, \dots, x_n]$ and $I = \langle f \rangle$ for some non-constant irreducible polynomial f. Give an explicit description of $\oplus_{d=0}^{\infty} I^d$ and $\oplus_{d=0}^{\infty} I^d/I^{d+1}$. (A good start is to construct a graded homomorphism $k[x_1, \dots, x_n, t] \to \oplus_{d=0}^{\infty} I^d$, with the grading given by t-degree, viewing $k[x_1, \dots, x_n, t]$ as polynomials in t with coefficients in $k[x_1, \dots, x_n]$.)

(10) Let $Y \subset \mathbb{P}^n$ be a projective variety. Prove that $\dim(Y) = \dim(S(Y)) - 1$.

(11) Let $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$ be a radical ideal and $Z = Z(\mathfrak{a})$ the algebraic set in $\mathbb{A}^n$ it defines. Consider the closure of Z in $\mathbb{P}^n$ under the map $(a_1, \dots, a_n) = (1 : a_1 : \cdots : a_n)$. Show that it is the zero set of the ideal $\langle F_f : f \in \mathfrak{a} \rangle$, where F_f denotes the homogenization of f relative to the additional variable x_0.

(12) Prove that the closure S of the affine surface S_0, given as $xy - z = 0$ in $\mathbb{A}^3_{x,y,z}$, in $\mathbb{P}^3_{x,y,z,w}$ is given by the equation $xy - zw$. Let $D = S - S_0$. The twisted cubic curve $\{(t, t^2, t^3) : t \in k\}$ lies on S_0. Determine how its closure in $\mathbb{P}^3$ intersects D. Further, let T_0 be the surface $xy - z^2$, T its closure in $\mathbb{P}^3$ and $E = T - T_0$. Calculate $T \cap S$ and $E \cap D$.

(13) Prove that the ring of regular functions of $U = \mathbb{A}^2 - \{(0, 0)\}$ is $k[x, y]$. Prove that U is not an affine variety.

(14) The Segre embedding. Let $\phi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \dots, a_r) \times (b_0, \dots, b_s)$ to $(\dots, a_i b_j, \dots)$ in lexicographic order, where $N = (r + 1)(s + 1) - 1$. Prove that ψ is well-defined and injective function. Show that the image of ψ is a subvariety of $\mathbb{P}^N$. (Hint: Let the homogenous coordinates on $\mathbb{P}^N$ be z_{ij}, $i = 0, \dots, r$, $j = 0, \dots, s$, and let $\mathfrak{a}$ be the kernel of the homomorphism $k[z_{ij}] \to k[x_0, \dots, x_r, y_0, \dots, y_s]$ that sends z_{ij} to $x_i y_j$. Show that $\text{Im}(\psi) = \mathcal{Z}(\mathfrak{a})$.)

(15) Let X be the curve in $\mathbb{A}^2$ given by $y^2 = x^3$. Let $f : \mathbb{A}^1 \to X$ be the morphism $f(t) = (t^2, t^3)$. Prove that f is bijective but not an isomorphism. Find the image of $k[X]$ in $k[t]$.

(16) Let X be the plane curve defined by $y^2 = x^3 + x^2$. Prove that $f : \mathbb{A}^1 \to X$, $f(t) = (t^2 - 1, t(t^2 - 1))$ is a morphism. Is it injective? surjective? isomorphism? Show that the image of $k[X]$ in $k[t]$ are the polynomials g such that $g(1) = g(-1)$.

(17) Prove that the hyperbola $xy = 1$ and the affine line $\mathbb{A}^1$ are not isomorphic.

(18) Consider the morphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ defined by $f(x, y) = (x, xy)$. Find the image $f(\mathbb{A}^2)$. is it open in $\mathbb{A}^2$? dense? closed?

(19) Prove that every automorphism of $\mathbb{A}^1$ is given by map of the form $f(x) = ax + b, a \neq 0$.

(20) Let $f : X \to Y$ be a morphism. The graph of f, Γ_f is the subset of $X \times Y$ consisting of the points $\{(x, f(x)) : x \in X\}$. Prove that it is a closed subset of $X \times Y$ that is isomorphic to X. Using this, prove that every morphism $f : X \to Y$ can be factored $p_Y \circ g$, where $p_Y : X \times Y \to Y$ is the projection and $g : X \to X \times Y$ is an embedding.

(21) In this exercise we prove that the product of varieties is a variety. Namely, given what was done in class, that it's irreducible.

Let X, Y be varieties.

- Show that for $y \in Y$, $X_y := X \times \{y\} \subset X \times Y$ is isomorphic to X.
- Let $X \times Y = Z_1 \cup Z_2$ a union of two closed sets. Using the irreducibility of X, prove that for each $y \in Y$, X_y is contained either in Z_1 or in Z_2 (or both). Prove that the set of points $y \in Y$ such that $X_y \subset Z_1$ is closed in Y. Similarly for Z_2.
- Using the irreducibility of Y, conclude the proof.
- Now, assuming that X, Y, are affine, deduce a statement about the tensor product of integral domains that are finitely generated k-algebras, k an algebraically closed field. (The assumptions are needed!)

(22) Let E be a d-dimensional linear subspace of $\mathbb{P}^n$ defined by the vanishing of $n - d$ linear equations $L_1 = \cdots = L_{n-d} = 0$. The projection with centre E is the rational map $\pi : \mathbb{P}^n \to \mathbb{P}^{n-d-1}$, $\pi(x) = (L_1(x) : \cdots : L_{n-d}(x))$. It is defined on $\mathbb{P}^n - E$.

If X is a subvariety of $\mathbb{P}^n$ disjoint from E then we get a morphism $\pi : X \to \mathbb{P}^{n-d-1}$. The geometric meaning of this projection is the following. Take any $n - d - 1$-dimensional linear subspace H of $\mathbb{P}^n$ that is disjoint from E. For every point $x \in \mathbb{P}^n - E$ there is a linear subspace $\langle E, x\rangle$ generated by E and x and $\langle E, x\rangle \cap H$ consists of one point. There is an isomorphism $H \cong \mathbb{P}^{n-d-1}$ under which this point is $\pi(x)$.

Apply all this to the situation of $E = (0 : 0 : 1)$ and $H = (x : y : 0)$. Given a conic $f(x, y, z)$ in $\mathbb{P}^2$ (namely, $f(x, y, z)$ is a quadratic irreducible homogenous polynomial passing through E), use this to show that $Z(f)$ is isomorphic to $\mathbb{P}^1$.

(23) Let f be a rational function on a variety X. Prove that there is a largest open set on which f is represented by a regular function.

(24) Let $f : X \dashrightarrow Y$ be a rational map. Prove that there is a largest open set U on which f is represented by a morphism.

(25) Let f be the rational function on $\mathbb{P}^2$ given by $f(x_0 : x_1 : x_2) = x_1/x_0$. Find the set of points where f is defined and describe the corresponding regular function.

Now think of f as a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. Find the set of points where f is defined and describe that morphism.

(26) Prove that the map $y_0 = x_1x_2, y_1 = x_0x_2, y_2 = x_0x_1$ defines a birational map f of $\mathbb{P}^2$ to itself. At which points are f and f^{-1} defined? What are the open sets mapped isomorphically by f?

(27) Assume that the characteristic of the base field is not 2. The Steiner surface S in $\mathbb{P}^3$ is defined by

$$x_1^2x_2^2 + x_2^2x_0^2 + x_0^2x_1^2 - x_0x_1x_2x_3 = 0.$$

Show that there is a rational morphism $\mathbb{P}^2 \to S$ given by

$$(u_0 : u_1 : u_2) \mapsto (u_1u_2 : u_0u_2 : u_0u_1 : u_0^2 + u_1^2 + u_2^2).$$

Is it a morphism? Is it birational? Is it bijective?

(28) Find the singular points of the Steiner surface. Calculate the tangent cone at the singular points and its nature (for example, irreducible smooth surface of degree d, a union of two transversely crossing smooth surfaces each of degree d, etc.)

(29) Show that every polynomial f in $k[x, y, z]$ can be written as $a(x^2 - y) - b(x^3 - z) + r$, where r is a polynomial in x alone and $a, b \in k[x, y, z]$. Let I be the ideal of the twisted cubic curve. Prove that if $f \in I$ then $r = 0$ and so $f = a(x^2 - y) - b(x^3 - z)$. This actually proves that $(x^2 - y, x^3 - z)$ is a basis for I. Write $z^2 - x^4y$ this way.

(30) Take the ideal $(x^2 - y, x^3 - z)$ and find a Gröbner basis for it relative to lex and invlex. For each of these orders determine the ideal $\langle \mathrm{LT}(I)\rangle$. (This is a bit of lengthy computation for lex; easy for invlex. Please do it by hand to get used to those computations.)

(31) Gröbner bases can be used to calculate projective closures. Let I be an ideal in $k[x_1, \dots, x_n]$ and let $g_1, \dots, g_s$ be a Gröbner basis for I relative to grlex. Homogenize the g_i relative to a new variable x_0. Prove that the homogenizations $G_1, \dots, G_s$ are a Gröbner basis for the homogenization of I relative the order on monomials given as follows $x^\alpha x_0^c > x^\beta x_0^d$ if $x^\alpha > x^\beta$ or $\alpha = \beta$ and $c > d$ (prove this is a monomial order); in particular, the provide equations for the closure of the affine algebraic set $Z(I)$ in the projective space $\mathbb{P}^n$, because this closure is just the projective algebraic set $Z(I^h)$. Denote this order by $<_{grlex+}$.

First prove the property that $\mathrm{LT}_{<_{grlex+}}(f^h) = \mathrm{LT}_{<_{grlex}}(f)$. Then show that the G_i are a Gröbner basis by showing that their leading terms generate the ideal $\langle LT((I^h))\rangle$ (namely, using the definition and not, say, Buchberger's criterion).

(32) Find a surface S in $\mathbb{A}^3$ with a unique singular point equal to $0 = (0, 0, 0)$ and such that the tangent cone at that point is $x_1x_2 = 0$. Blow up the surface S at the point 0 and calculate the singular points of the blow-up.

(33) Assume that we are in characteristic zero (to avoid considering special cases). Consider the curve $Y_{a,b} : y^a = x^b$, where a, b are integers, both greater than 1 ($a = b$ is allowed and is in a certain sense a special case). Prove that it has a unique singular point and calculate the tangent space and tangent cone at that point. Blow-up the curve $Y_{a,b}$ at the point $(0, 0)$.

Calculate the singularities of $\widetilde{Y}_{a,b}$. At each singular point calculate the dimension of the tangent space and the tangent cone. In which cases the singularities were resolved?

(34) Assume that we are in characteristic zero (to avoid considering special cases). Calculate the singular points of "the pince" $S : xy^2 = z^2$ in $\mathbb{A}^3$; calculate their tangent spaces and tangent cones. Blow-up S at the point $(0,0)$. Calculate the singular points of the blow-up $\widetilde{S}$. Calculate the tangent cone and the dimension of the tangent space at the points of the exceptional fibre.

(35) Assume that we are in characteristic zero (to avoid considering special cases). Does blow-up resolves the singularities of the plane curve $Y : x^2y + xy^2 = x^4 + y^4$?

(36) Let k be an algebraically closed field of characteristic different than 3. The group μ_3, the roots of unity of order 3 acts on the ring of polynomials $k[x,y]$

$$[\alpha](x) = \alpha \cdot x, \quad [\alpha](y) = \alpha^2 \cdot y$$

(and extend by linearity and multiplicativity). Let A be the ring of invariants. Let X be the affine variety with $A \cong A(X)$. Realize X as an affine variety in some affine space. Show that the inclusion $A(X) \to k[x,y]$ gives a finite morphism $\mathbb{A}^2 \to X$ and write it down explicitly. Find a finite morphism $X \to \mathbb{A}^2$ by using Noether's normalization lemma.