NOTES FOR THE COURSE IN ALGEBRAIC GEOMETRY, PART II, WINTER 2015

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Contents

1. Differentials and regularity

1.1. **Derivations and the module of differentials - affine theory.**

A = ring

B = A-algebra (where the ring homomorphism $A \longrightarrow B$ need not be injective)

M = B module

Definition 1.1.1. An A-derivation $d : B \longrightarrow M$ is a function that is:

(1) additive;
(2) $d(b_1 b_2) = b_1 \cdot db_2 + b_2 \cdot db_1$;
(3) $da = 0, \forall a \in A$.

We shall denote the set of A-derivations valued in M by $\mathrm{Der}_A(B, M)$.

Remarks: First, note that the B-module structure is used to express the Leibniz rule (ii), but the function d is not required to be B-linear. In fact, if it were, d would be identically zero, as the following argument shows.

First note that $d1 = d(1 \cdot 1) = 2 \cdot d1$ and thus $d1 = 0$. Condition (iii) is equivalent to requiring that d is A-linear. Indeed, assume d is A-linear then $da = a \cdot d1 = a \cdot 0 = 0$. Conversely, if $da = 0$ for all $a \in A$ then $d(ab) = a \cdot db + b \cdot da = a \cdot db$.

We now seek an A-derivation that is universal. Namely a B-module, to be denoted $\Omega_{B/A}$ and an A-derivation

$$d = d_{B/A} : B \longrightarrow \Omega_{B/A},$$

such that for every B-module M and a derivation $\delta : B \longrightarrow M$ there is a B-module homomorphism $f : \Omega_{B/A} \longrightarrow M$ making the following diagram commutative:

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "\delta"'] & \Omega_{B/A} \arrow[d, "f"] \\ & M. \end{tikzcd}$$

As usual, if $(d, \Omega_{B/A})$ exists, it is unique up to unique isomorphism.

Proposition 1.1.2. *$(d, \Omega_{B/A})$ exists. It is called the module of relative differentials of B over A.*

Proof. This is an easy formal construction. Define $\Omega_{B/A}$ to be the quotient of the free B module $\oplus_{b \in B} B \cdot db$ by the B submodule generated by all the expressions $\{da : a \in A\}$, $\{d(b_1 b_2) - b_1 \cdot db_2 - b_2 \cdot db_1 : b_1, b_2 \in B\}$, $\{d(b_1 + b_2) - db_1 - db_2\}$. Further, define $d : B \longrightarrow \Omega_{B/A}$ by sending b to db.

Given a derivation $\delta : B \longrightarrow M$ define a map $f : \Omega_{B/A} \longrightarrow M$ by $f(\sum_i \beta_i db_i) = \sum_i \beta_i \delta b_i$. It is immediate to check this is well-defined homomorphism of B module that satisfies $f \circ d = \delta$. $\square$

Corollary 1.1.3. *$\Omega_{B/A}$ is generated as a B-module by the set $\{db : b \in B\}$. For every B-module T, $\mathrm{Hom}_B(\Omega_{B/A}, T) = \mathrm{Der}_A(B, T)$.*

Proposition 1.1.4 (The diagonal)**.** *Let $f : B \otimes_A B \longrightarrow B$ be the multiplication map $f(b_1 \otimes b_2) = b_1 b_2$. Let I be the kernel of f. Consider $B \otimes_A B, I$ and I^2 as left B-modules. Define a map*

$$\delta : B \longrightarrow I/I^2, \quad \delta b = 1 \otimes b - b \otimes 1.$$

Then $(\delta, I/I^2)$ is a universal module of differentials for B over A. Thus, $(\delta, I/I^2) \cong (d_{B/A}, \Omega_{B/A})$.

Proof. We shall define an isomorphism $\Omega_{B/A} \longrightarrow I/I^2$ that commutes with the derivations. First, we verify that δ is indeed a derivation. Additivity and A-linearity are clear. Furthermore,

$$\begin{aligned} \delta(b_1 b_2) - b_1 \delta b_2 - b_2 \delta b_1 &= (1 \otimes b_1 b_2 - b_1 b_2 \otimes 1) - (b_1 \otimes b_2 - b_1 b_2 \otimes 1) - (b_2 \otimes b_1 - b_1 b_2 \otimes 1) \\ &= 1 \otimes b_1 b_2 + b_1 b_2 \otimes 1 - b_1 \otimes b_2 - b_2 \otimes b_1 \\ &= (1 \otimes b_1 - b_1 \otimes 1)(1 \otimes b_2 - b_2 \otimes 1) \in I^2. \end{aligned}$$

Next, the universal property of $\Omega_{B/A}$ gives us a map of B-modules, commuting with the derivation maps,

$$\Omega_{B/A} \longrightarrow I/I^2, \qquad db \mapsto 1 \otimes b - b \otimes 1.$$

We need to construct an inverse to this map. To do so, consider the B-module

$$B \oplus \Omega_{B/A},$$

and define on it the operation

$$(b_1, m_1) \cdot (b_2, m_2) = (b_1 b_2, b_1 m_2 + b_2 m_1).$$

With some patience one verifies that this gives a commutative ring structure and, evidently the ideal $\Omega_{B/A}$ satisfies that $(\Omega_{B/A})^2 = \{0\}$ in this ring. We define a ring homomorphism

$$B \otimes_A B \longrightarrow B \oplus \Omega_{B/A}, \quad b_1 \otimes b_2 \mapsto (b_1 b_2, b_1 db_2).$$

(There is a clear lack of symmetry here, but we had already broken the symmetry by deciding that $B \otimes_A B$ is considered as a left B-module.) It is straight-forward to verify that this is indeed a ring homomorphism. Indeed, additivity is easy and 1 goes to $(1, 0)$ which is the unit element of $B \oplus \Omega_{B/A}$. As to multiplication, the key calculation is that $(b_1 \otimes b_2)(b_1' \otimes b_2') = b_1 b_1' \otimes b_2 b_2'$ is mapped to $(b_1 b_1' b_2 b_2', b_1 b_1' b_2 db_2' + b_1 b_1' b_2' db_2)$, which is indeed the product $(b_1 b_2, b_1 db_2)(b_1' b_2', b_1' db_2')$. Finally, this is also a map of B-modules as $b(b_1 \otimes b_2) = bb_1 \otimes b_2 \mapsto (bb_1 b_2, bb_1 db_2) = b(b_1 b_2, b_1 db_2)$.

Now, the induced map on I takes I to $\{0\} \times \Omega_{B/A} \subset B \oplus \Omega_{B/A}$. As $\Omega_{B/A}^2 = 0$, we get a well-defined homomorphism of B modules

$$I/I^2 \mapsto \Omega_{B/A}, \quad \sum b_i \otimes b_i' \mapsto (0, \sum b_i db_i').$$

Now, the map $\Omega_{B/A} \longrightarrow I/I^2$ takes db to $\delta b = 1 \otimes b - b \otimes 1$, which goes under the map $I/I^2 \mapsto \Omega_{B/A}$ above to $(b, db) - (b, 0) = (0, db)$. That is, the composition $\Omega_{B/A} \longrightarrow I/I^2 \longrightarrow \Omega_{B/A}$ is the identity. As we have seen last term I/I^2 is generated as a B-module by the elements $b_1 \otimes b_2 - b_2 \otimes b_1$.

Such an element is mapped to $(b_1b_2, b_1db_2) - (b_1b_2, b_2db_1) = (0, b_1db_2 - b_2db_1)$ in $B \oplus \Omega_{B/A}$ that is then mapped to $b_1\delta b_2 - b_2\delta b_1 = b_1(1\otimes b_2 - b_2\otimes 1) - b_2(1\otimes b_1 - b_1 \otimes 1)$. But this is just $b_1\otimes b_2 - b_2\otimes b_1$ again. Thus, also the composition $I/I^2 \longrightarrow \Omega_{B/A} \longrightarrow I/I^2$ is the identity and the proof is complete. $\square$

Remark 1.1.5. Let $B \longrightarrow C$ be a homomorphism of rings. We shall often use the canonical isomorphism $\mathrm{Hom}_C(X\otimes_B C, Y) = \mathrm{Hom}_B(X, Y)$, X a B-module, Y a C-module. This isomorphism expresses the fact that the pair of functors $((-)\otimes_B C, F)$, between B-modules and C-modules (where F is the forgetful functor) is an adjoint pair. More generally, for a C-module M, we have $\mathrm{Hom}_C(X\otimes_B M, Y) = \mathrm{Hom}_B(X, \mathrm{Hom}_C(M, Y))$, which is the adjoint property of the functors $((-)\otimes_B M, \mathrm{Hom}_C(M, -))$. (Note that for $M = C$ this gives back the previous adjoint pair).

Proposition 1.1.6 (Exercise 1; base change and localization properties)**.** *(1) Let A_1 be an A-algebra and define $B_1 = A_1\otimes_A B$, which is an A_1-algbera and a B-algebra. Then*

$$\Omega_{B_1/A_1} = B_1 \otimes_B \Omega_{B/A}.$$

(2) Let S be a multiplicative set in B then

$$\Omega_{B[S^{-1}]/A} = B[S^{-1}]\otimes_B \Omega_{B/A}.$$

A geometric moment: The homomorphism of rings $A \longrightarrow B$ corresponds to a morphism of schemes $\mathrm{Spec}\ B \longrightarrow \mathrm{Spec}\ A$. The ideal I is the ideal defining the closed immersion

$$\Delta : \mathrm{Spec}\ B \longrightarrow \mathrm{Spec}\ B \times_{\mathrm{Spec}\ A} \mathrm{Spec}\ B = \mathrm{Spec}(B\otimes_A B).$$

More precisely, the associated quasi-coherent sheaf $\tilde{I}$ corresponds to that closed immersion. $\tilde{I}/\tilde{I}^2$ is initially a sheaf of modules on $\mathrm{Spec}(B\otimes_A B)$. On the other hand I/I^2 viewed as a B-module (as above) define a quasi-coherent sheaf $\tilde{I}/\tilde{I}^2$ also on $\mathrm{Spec}\ B$. The relation between those two sheaves is that

$$\Delta_*(\tilde{I}/\tilde{I}^2) = \tilde{I}/\tilde{I}^2 = \widetilde{I/I^2}.$$

(Which allows for the ambiguity in notation.) Thus, the algebraic construction of the module of differentials $\Omega_{B/A}$ finds a geometric interpretation as the quasi coherent module I/I^2 coming from the diagonal morphism $\Delta : \mathrm{Spec}\ B \longrightarrow \mathrm{Spec}\ B\otimes_A B$. The content of the preceding proposition is that these constructions commute with base-change and localization. This is almost immediate from the geometric interpretation, but it is perhaps healthier to give a direct algebraic proof.

1.2. **Exact sequences and key examples.**

Proposition 1.2.1 (1st Exact sequence). *Let $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ be ring homomorphisms. There is an exact sequence of C-modules,*

$$\Omega_{B/A} \otimes_B C \xrightarrow{v} \Omega_{C/A} \xrightarrow{u} \Omega_{C/B} \longrightarrow 0.$$

Furthermore, v is injective and its image a direct summand if and only if any A-derivation $B \longrightarrow M$ into a C-module M can be extended to an A-derivation $C \longrightarrow M$.

Proof. The maps v and u are natural, determined by $v(d_{B/A}b \otimes c) = c \cdot d_{C/A}\psi(b)$ and $u(c \cdot d_{C/A}c_1) = c \cdot d_{C/B}c_1$. Note that $u \circ v(d_{B/A}b \otimes c) = c \cdot d_{C/B}\psi(b) = c \cdot 0 = 0$ and, thus, $u \circ v = 0$. Furthermore, as $\Omega_{C/B}$ is generated as a C module by the symbols $\{dc : c \in C\}$, the surjectivity of u is clear. It remains to show that $\mathrm{Im}(v) = \mathrm{Ker}(u)$. We use the following lemma.

Lemma 1.2.2 (Exercise 2). *To show that a complex of C-modules,*

$$M_1 \xrightarrow{v} M_2 \xrightarrow{u} M_3 ,$$

is exact, it suffices[1] *to show that for every C-module T, the following sequence is exact:*

$$\mathrm{Hom}_C(M_1, T) \xleftarrow{v^*} \mathrm{Hom}_C(M_2, T) \xleftarrow{u^*} \mathrm{Hom}_C(M_3, T) .$$

We make use of the lemma in our situation; we need to show exactness of the upper row

$$\begin{tikzcd}
\mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, T) \arrow[d, equal] & \mathrm{Hom}_C(\Omega_{C/A}, T) \arrow[l, "v^*"'] \arrow[dd, equal] & \mathrm{Hom}_C(\Omega_{C/B}, T) \arrow[l, "u^*"'] \arrow[dd, equal] \\
\mathrm{Hom}_B(\Omega_{B/A}, T) \arrow[d, equal] & & \\
\mathrm{Der}_A(B, T) & \mathrm{Der}_A(C, T) \arrow[l, "v^*"'] & \mathrm{Der}_B(C, T). \arrow[l, "u^*"']
\end{tikzcd}$$

The exactness of the bottom row follows directly from the definitions.

For the last part we use the following lemma.

Lemma 1.2.3 (Exercise 3). *A homomorphism $v : M_1 \longrightarrow M_2$ is injective and its image a direct summand, if and only if the homorphism $v^* : \mathrm{Hom}_C(M_2, T) \longrightarrow \mathrm{Hom}_C(M_1, T)$ is surjective for all C-modules T.*

We apply the lemma to the map $v : \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A}$; we need thus to consider the upper row of the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, T) \arrow[d, equal] & \mathrm{Hom}_C(\Omega_{C/A}, T) \arrow[l, "v^*"] \arrow[d, equal] \\
\mathrm{Der}_A(B, T) & \mathrm{Der}_A(C, A) \arrow[l, "v^*"]
\end{tikzcd}$$

[1] But not necessary! Consider $0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ and $T = \mathbb{Z}$.

The bottom row gives the reinterpretation of the surjectiveness of v^* appearing in the statement of the theorem. ☐

Proposition 1.2.4 (2nd Exact sequence)**.** *Let B be an A-algebra and $C = B/I$ for some ideal I of B. There is an exact sequence of C-modules,*

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow 0,$$

where $\delta \bar{b} = db \otimes 1$.

Remark 1.2.5. Note that $\Omega_{C/B} = 0$. Thus, the 2nd exact sequence improves on the first exact sequence by determining the kernel of $\Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A}$.

Proof. We first remark that I/I^2 is a B-module killed by I, hence naturally a C-module. The map δ is well-defined: we first define it as a map $I \longrightarrow \Omega_{B/A} \otimes_B C$ and check that it kills I^2. Indeed, $\delta(b_1 b_2) = b_1 db_2 \otimes 1 + b_2 db_1 \otimes 1 = db_2 \otimes b_1 + db_1 \otimes b_2$ and this is zero because both b_i (being elements of I) are zero in C. The function δ is indeed C-linear. Additivity is clear. Next, $\delta(\bar{c}\bar{b}) = d(cb) \otimes 1 = cdb \otimes 1 + bdc \otimes 1 = c(db \otimes 1) = c \cdot \delta\bar{b}$.

Now, it follows directly from the definitions that we have a complex

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A},$$

and so, using Lemma 1.2.2, we need to show the exactness of the complex

$$\mathrm{Der}_A(C, T) \longrightarrow \mathrm{Der}_A(B, T) \longrightarrow \mathrm{Hom}_C(I/I^2, T),$$

where the first map is restriction of derivations and the second map takes a derivation $d : B \longrightarrow T$ to the C-module homomorphism $I/I^2 \longrightarrow T$ that takes $\bar{b} \mapsto db$. Now, the only statement left to check is that an A-derivation of B that takes I to zero is a derivation of C, but that is clear. ☐

Proposition 1.2.6. *Let A be a ring, $A[\underline{x}] := A[x_1, \dots, x_n]$ the ring of polynomials in n variables over A, then*

$$\Omega_{A[\underline{x}]/A} = \oplus_{i=1}^n A[\underline{x}] \cdot dx_i.$$

More generally, let $\{f_1, \dots, f_m\} \subset A[\underline{x}]$ and $C = A[x_1, \dots, x_n]/\langle f_1, \dots, f_m\rangle$, then

$$\Omega_{C/A} = \oplus_{i=1}^n C \cdot dx_i / \langle \{\sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \cdot dx_j : i = 1, \dots, m\} \rangle_C.$$

Proof. First, for any $f \in A[\underline{x}]$ we have the identity in $\Omega_{A[\underline{x}]/A}$,

$$df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \cdot dx_j.$$

This identity is A-linear in f, so to verify it one can assume that $f = x_1^{a_1} \cdots x_n^{a_n}$ and proceed by induction on the total degree of f. It is easy to check by induction on $a_1 \geq 0$ that $dx_1^{a_1} = a_1 x_1^{a_1 - 1} dx_1$.

Now, if $a_1 > 0$ then $f = x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n} = x_1^{a_1} \cdot g$ and thus, using the induction hypothesis for g, $df = x_1^{a_1} \sum_{j=2}^n \frac{\partial g}{\partial x_j} \cdot dx_j + g \cdot a_1 x_1^{a_1 - 1} dx_1 = \sum_{j=2}^n \frac{\partial f}{\partial x_j} \cdot dx_j + a_1 x_1^{a_1-1} x_2^{a_2} \cdots x_n^{a_n} dx_1 = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \cdot dx_j$.

It follows that $\Omega_{A[\underline{x}]/A}$ is generated as an $A[\underline{x}]$ module by $dx_1, \dots, dx_n$. Suppose that there is a relation between these differentials:

$$\sum_{i=1}^n P_i dx_i = 0,$$

for some $P_i \in A[\underline{x}]$. As $\frac{\partial}{\partial x_i}$ is an A-derivation of $A[\underline{x}]$ valued in $A[\underline{x}]$, there is an $A[\underline{x}]$-module homomorphism $f_i : \Omega_{A[\underline{x}]/A} \longrightarrow A[\underline{x}]$ such that the following diagram commutes:

$$\begin{tikzcd} A[\underline{x}] \arrow[r, "d"] \arrow[dr, "\frac{\partial}{\partial x_i}"'] & \Omega_{A[\underline{x}]/A} \arrow[d, "f_i"] \\ & A[\underline{x}]. \end{tikzcd}$$

Under this maps we have $x_j \mapsto \delta_{ij} \cdot dx_j$ and $\partial x_j / \partial x_i = \delta_{ij}$ (Kronecker's δ). It follows that $f_i(dx_j) = \delta_{ij}$ and so that $0 = f_i(\sum_{j=1}^n P_j dx_j) = \sum_{j=1}^n P_j f_i(dx_j) = P_i$ (where we have used strongly that f_i is a homomorphism of $A[\underline{x}]$-modules). Thus, each $P_i = 0$.

To get the stronger claim, use Proposition 1.2.4, where $I = \langle f_1, \dots, f_m \rangle$. We have,

$$\begin{aligned} \Omega_{C/A} &= \left(\Omega_{A[\underline{x}]/A} \otimes_{A[\underline{x}]} C\right) / \delta(I/I^2) \\ &\cong \left(\oplus_{i=1}^n C \cdot dx_i\right) / \delta(I/I^2), \end{aligned}$$

but, as I/I^2 is generated as a C module by $f_1, \dots, f_m$ and $\delta(\bar{f}_i) = df_i \otimes 1 = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \cdot dx_j$ we find that $\delta(I/I^2) = \langle \{ \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \cdot dx_j : i = 1, \dots, m \} \rangle_C$. □

Example 1.2.7. We provide some simple examples. More to follow!

(1) We first remark that Proposition 1.2.6 holds equally well for infinite number of variables x_i and polynomials f_j. However, to simplify, we would often just consider the finitely generated case. For example:

(2) Let K/F be a finite separable extension of a fields. Without loss of generality, $K = F[t]/(f(t))$, where f is an irreducible separable polynomial of $F[t]$. Thus, $f'(t)$ is non-zero in K. Consequently, $\Omega_{K/F} = F[t] \cdot dt / F[t] \cdot f'(t) \cdot dt = \{0\}$.

(3) Consider now the case of a purely transcendental extension $K = F(t)$. First, $\Omega_{F[t]/F} = F[t] \cdot dt$. Using localization we find that

$$\Omega_{F(t)/F} = F(t) \cdot dt.$$

In particular, the canonical map $f : \Omega_{F(t)/F} \longrightarrow F(t)$, taking $g(t) \cdot dt$ to $g(t)$ provides us with an F-derivation $F(t) \longrightarrow F(t)$ for which $t \mapsto f(dt) = 1$. This is no-surprise; this is simply the derivation $\frac{\partial}{\partial t}$.

(4) Consider now that following case: F is a field of characteristic p, $a \in F$ is not a p-power and $K = F[t]/(t^p - a)$. Then K is a field and, as $d_{K/F}(t^p - a) = pt^{p-1}dt = 0$,
$$\Omega_{K/F} = K \cdot dt.$$
We can therefore repeat the argument made in the previous example to deduce that there is an F-derivation $\delta : K \longrightarrow K$ such that $\delta(t) = 1$.

(5) Consider the cuspidal curve $B = k[x, y]/(y^2 - x^3)$ over k. The B-module $\Omega_{B/k}$ has torsion elements. Indeed, consider $2x \cdot dy - 3y \cdot dx$, which is a non-zero element of $\Omega_{B/k}$. We find that $y(2x \cdot dy - 3y \cdot dx) = 2xy \cdot dy - 3x^3 \cdot dx = x(d(y^2 - x^3)) = 0$. This is of interest because we shall later prove that if we have a non-singular variety over a field k, for example a non-singular affine variety Spec(A), where A is a finitely-generated k-algebra that is a domain, then the sheaf of relative differentials $\Omega_{A/k}$ is locally free and in particular a torsion-free A-module. Thus, our example shows on the level of differentials that Spec(B) is singular.

It will be very useful to us to have an even stronger form of the last proposition. The proof is similar and is left as an exercise.

Proposition 1.2.8 (Exercise 4)**.** *Let k be a ring and A a k-algebra. Then:*

(1) $\Omega_{A[\underline{x}]/k} = (\Omega_{A/k} \otimes_A A[\underline{x}]) \oplus \oplus_{i=1}^n A[\underline{x}] \cdot dx_i$ (the canonical isomorphism being induced by Proposition 1.2.1).

(2) Let $\mathfrak{m} = \langle f_1, \dots, f_m\rangle$ be an ideal of $A[\underline{x}]$ and let $C = A[\underline{x}]/\mathfrak{m}$. Show that
$$\Omega_{C/k} \cong (\Omega_{A/k} \otimes_A (A[\underline{x}]/\mathfrak{m})) \oplus \oplus_{i=1}^n (A[\underline{x}]/\mathfrak{m}) \cdot dx_i,$$
modulo $\delta(\mathfrak{m}/\mathfrak{m}^2)$, where
$$\delta(f) = (d_0 f)(x) + \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i,$$
and where for $f = \sum_I a_I x^I$, $a_I \in A$ we let
$$(d_0 f)(x) = \sum_I d_{A/k} a_I (\mathrm{mod}\ \mathfrak{m}) \cdot x^I.$$

1.3. **Differentials and separability.** Let k be a field and $K = k(t)$ a purely transcendental extension. We have $\Omega_{k[t]/k} = k[t] \cdot dt$ and so, by localization, $\Omega_{K/k} = K \cdot dt$. More generally,

(I) If $k \subset L$ are fields and $K = L(t)$ then $\Omega_{L[t]/k} = (\Omega_{L/k} \otimes_L L[t]) \oplus \Omega_{L[t]/L}$. Localizing, we obtain
$$\Omega_{K/k} = \Omega_{L/k} \otimes_L K \oplus K \cdot dt.$$
We note that $\Omega_{L/k}$ is a vector space over L and we conclude that
$$\dim_K(\Omega_{K/k}) = \dim_L(\Omega_{L/k}) + 1, \qquad K = L(t).$$

(II) If $K = L[t]/f$ where f is a separable and irreducible polynomial in $L[t]$ then we find, using Proposition 1.2.8, that

$$\Omega_{K/k} = \left(\Omega_{L/k} \otimes_L K \oplus K \cdot dt\right) / \langle \delta f \rangle_{L[t]},$$

where $\delta f = (d_0 f)(t) + \frac{\partial f}{\partial t} dt$. As $(f, \frac{\partial f}{\partial t}) = 1$ we have $\frac{\partial f}{\partial t}$ invertible in K and it follows that $\Omega_{K/k} = \Omega_{L/k} \otimes_L K$. In this case,

$$\dim_K(\Omega_{K/k}) = \dim_L(\Omega_{L/k}), \quad K/L \text{ finite separable.}$$

We recall the notion of a separating transcendence basis. Let K/k be an extension of fields. A transcendence basis $\{t_\alpha\}$ is called a separating transcendence basis if the extension $K/k(\{t_\alpha\})$ is (algebraic and) separable. If k is a perfect field, any extension K/k has a separating transcendence basis. Our discussion so far gives the following.

Corollary 1.3.1. *Let K/k be a finitely generated field extension that has a separating transcendence basis. Then,*

$$\dim_K(\Omega_{K/k}) = \text{tr. deg.}(K/k).$$

Now, in considering how field extensions are built[2], there are two more possibilities to consider. They only need to be considered in characteristic p.

(III) $k \subset L \subset K$ an extension of fields of characteristic p, $K = L[t]/(t^p - a)$ where $a \in L$. In this case, in the notation of Proposition 1.2.8, $\delta(t^p - a) = d_{L/k}(a)$. There are two possibilities. If $d_{L/k}(a) = 0$ then

$$\Omega_{K/k} = \Omega_{L/k} \otimes_L K \oplus K \cdot dt.$$

Thus,

$$\dim_K(\Omega_{K/k}) = \dim_L(\Omega_{L/k}) + 1, \qquad K = L[t]/(t^p - a), d_{L/k}(a) = 0.$$

(IV) The remaining case is when $d_{L/k}(a) \neq 0$ (and the rest is as above). In this case we find that

$$\dim_K(\Omega_{K/k}) = \dim_L(\Omega_{L/k}) > 0.$$

Our goal is to prove now that if $\dim_K(\Omega_{K/k}) = \text{tr. deg.}(K/k)$, and K is finitely generated over k, then K has a separating transcendence basis over k. As K can be obtained from k by iterating constructions as in **(I) - (IV)**, we may conclude that if for a subfield L/k we have $\Omega_{L/k} \neq 0$ then also $\Omega_{K/k} \neq 0$.

[2]Any Field extension K/L has the following filtration $L \subseteq L_1 \subseteq L_2 \subseteq K$, where L_1/L is a purely transcendental extension, L_2/L_1 is an algebraic separable extension and K/L_2 is an algebraic purely inseparable extension, which means that every element of K not in L_2 is not a root of separable polynomial with coefficients in L_2. If K/L_2 is a finitely generated non-trivial extension, then $[K : L_2] = p^a$, $a > 0$, and p is a prime equal to the characteristic of L.

Said differently, if we assume first that $\Omega_{K/k} = \{0\}$ then it follows that for every subfield $k \subseteq L \subseteq K$ we have $\Omega_{L/k} = \{0\}$. In this case we find that K is obtained from k using only construction (**II**), repeatedly. Thus, K/k is a finite separable extension.

Now suppose that $\dim_K(\Omega_{K/k}) = r$ and choose elements $x_1, \dots, x_r$ such that $dx_1, \dots, dx_r$ are a basis for the K-vector space $\Omega_{K/k}$. Let $L = k(x_1, \dots, x_r)$. We have the exact sequence of Proposition 1.2.1

$$\Omega_{L/k} \otimes_L K \longrightarrow \Omega_{K/k} \longrightarrow \Omega_{K/L} \longrightarrow 0.$$

As all dx_i are in the image of the first arrow, the first arrow is surjective and consequently $\Omega_{K/L} = \{0\}$. By the previous case treated, K is separable over L. Because K and L have the same transcendence degree over k, and the transcendence degree of L over k is r if and only if $\{x_1, \dots, x_r\}$ are a transcendence basis, it follows that $\{x_1, \dots, x_r\}$ are a separating transcendence basis for K over k.

In summary, we have proved the following theorem.

Theorem 1.3.2. *Let K/k be a finitely generated field extension. Then K/k has a separating transcendence basis if and only if $\dim_K(\Omega_{K/k}) = \mathrm{tr.\,deg.}(K/k)$. If this equality holds, and say $r = \dim_K(\Omega_{K/k})$, any subset $\{x_1, \dots, x_r\} \subset K$ such that $\{dx_1, \dots, dx_r\}$ are a basis for $\Omega_{K/k}$ over K is a separating transcendence basis for K/k.*

1.4. **Differentials for local rings.** In this section we consider the differentials of a local ring $(B, \mathfrak{m})$ under the assumption that there is a map $k := B/\mathfrak{m} \longrightarrow B$ such that the composition

$$k \longrightarrow B \longrightarrow B/\mathfrak{m} = k, \tag{1}$$

is the identity map. This situation is typical for varieties V over an algebraically closed field k. If v is a closed point of V then $\boldsymbol{k}(v)$ is naturally isomorphic to k.

Theorem 1.4.1. *Let B be a local ring containing a field k as above (1), then*

$$\Omega_{B/k} \otimes_B k \cong \mathfrak{m}/\mathfrak{m}^2.$$

Proof. From Proposition 1.2.4

$$\begin{tikzcd} \mathfrak{m}/\mathfrak{m}^2 \arrow[r, "\delta"] & \Omega_{B/k} \otimes_B k \arrow[r] & \Omega_{k/k} \arrow[r] \arrow[d, equal] & 0 \\ & & 0 & \end{tikzcd}$$

Thus, δ is surjective. To show δ is injective, we need to show that the dual map δ^* is surjective, where

$$\delta^* : \mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k) \longrightarrow \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k).$$

However, $\mathrm{Hom}_k(\Omega_{B/k} \otimes_B k, k) = \mathrm{Hom}_B(\Omega_{B/k}, k) = \mathrm{Der}_k(B, k)$. Given $h \in \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ define a derivation $d_h \in \mathrm{Der}_k(B, k)$ in the following way. First, note that $B = k \oplus \mathfrak{m}$ and so we can write any element of B as $\lambda + m, \lambda \in k, m \in \mathfrak{m}$. Define then

$$d_h(\lambda + m) = h(\bar{m}).$$

We claim that this is a k-derivation. Additivity is clear. Now, $(\lambda_1 + m_1)(\lambda_2 + m_2) = \lambda_1\lambda_2 + (\lambda_1 m_2 + \lambda_2 m_1) + m_1 m_2$ and so

$$\begin{aligned} d_h((\lambda_1 + m_1)(\lambda_2 + m_2)) &= h(\lambda_1 m_2 + \lambda_2 m_1 + m_1 m_2) \\ &= \lambda_1 h(m_2) + \lambda_2 h(m_1) \\ &= (\lambda_1 + m_1)h(m_2) + (\lambda_2 + m_2)h(m_1) \\ &= (\lambda_1 + m_1)d_h(\lambda_2 + m_2) + (\lambda_2 + m_2)d_h(\lambda_1 + m_1). \end{aligned}$$

To verify it is a k-derivation, we only need to check that $d_h(\lambda) = 0, \lambda \in k$, but that is clear. Finally, we check that $\delta^*(d_h) = h$.

The derivation we defined comes from the homomorphism $\Omega_{B/k} \otimes_B k \longrightarrow k$ that has the property that $d\lambda \otimes 1 \mapsto 0, \lambda \in k$ and $dm \otimes 1 = h(\bar{m})$ for $m \in \mathfrak{m}$. The map $\delta : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{B/k} \otimes_B k$ takes m to $dm \otimes 1$ and so the induced homomorphism $\mathfrak{m}/\mathfrak{m}^2 \longrightarrow k$ is $m \mapsto \delta(m) = dm \otimes 1 \mapsto h(m)$. $\square$

Example 1.4.2. Let k be an algebraically closed field. Suppose that V is a variety defined in $\mathbb{A}^n_k$ by the polynomials $\{f_1, \dots f_m\}$. Namely, in classical algebraic geometry V is the collection of n-tuples $(v_1, \dots, v_n)$ such that $f_i(v_1, \dots, v_n) = 0$ for all $i = 1, \dots, m$. Suppose (and that can always be achieved by change of coordinates) that $\underline{0} = (0, \dots, 0) \in V$. Let us consider the maximal ideal at the point $\underline{0}$.

Denote by $f_{i,1}$ the linear term of f_i. Note that $f_i \equiv f_{i,1} \pmod{\mathfrak{m}^2}$. We therefore find that

$$\mathfrak{m}/\mathfrak{m}^2 = \oplus_{i=1}^n k x_i / \langle f_{1,1}, \dots, f_{m,1}\rangle_k.$$

Thus,

$$\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) = \{(v_1, \dots, v_n) : f_{j,1}(v_1, \dots, v_n) = 0, j = 1, \dots, m\}.$$

Note, however, that

$$f_{j,1}(v_1, \dots, v_n) = \langle (\frac{\partial f_j}{\partial x_1}(\underline{0}), \dots, \frac{\partial f_j}{\partial x_n}(\underline{0})), (v_1, \dots, v_n)\rangle,$$

where the brackets on the right are "inner product". Another way to phrase this calculation is that the Zariski tangent space we have studied in the last term (i.e., $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ in our case) can in this case be interpreted as the linear subspace of $\mathbb{A}^n_k$ defined by the linear equations

$$\sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(\underline{0}) \cdot x_i, \quad j = 1, \dots, m.$$

If k is the field of complex numbers, we recognize these equations as the equations for the tangent vectors to V at $\underline{0}$. This strengthens our interpretation of $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ as the tangent space at the point x corresponding to the maximal ideal $\mathfrak{m}$.

The moral of the story is that we have a sheaf $\tilde{\Omega}_{B/k}$, at this point over $\mathrm{Spec}(B)$ but soon in general, that deserves the name the cotangent sheaf. It has the property that its reduction modulo $\mathfrak{m}$, namely, $\Omega_{B/k} \otimes_B k$ is the dual of the Zariski tangent space.

1.5. **Globalization.** We have so far defined a B-module $\Omega_{B/A}$ for a ring homomorphism $A \longrightarrow B$. Through the method of constructing quasi-coherent sheaves we thus have a quasi coherent module $\tilde{\Omega}_{B/A}$ for a morphism $\mathrm{Spec} B \longrightarrow \mathrm{Spec} A$. It behaves "sensibly" relative to base-change and localization (Proposition 1.1.6). We would now like to extend our construction to any morphism $f : X \longrightarrow Y$ of schemes, making use of Proposition 1.1.4.

Let $f : X \longrightarrow Y$ be a morphism of schemes and $\Delta : X \longrightarrow X \times_Y X$ the diagonal morphism. Recall that $\Delta(X)$ is a locally closed subscheme of $X \times_Y X$. Namely, there is an open subscheme W of $X \times_Y X$ in which $\Delta(X)$ is a closed subscheme defined by a quasi-coherent sheaf of ideals $\mathscr{I}$[3].

Definition 1.5.1. We define

$$\Omega_{X/Y} = \Delta^*(\mathscr{I}/\mathscr{I}^2).$$

Note that $\Omega_{X/Y}$ is quasi-coherent $\mathcal{O}_X \cong \mathcal{O}_{\Delta(X)}$-module and if $X = \mathrm{Spec}\ B, Y = \mathrm{Spec}\ A$ then $\Omega_{X/Y} \cong \Omega_{B/A}$. The following results follow from their affine counterparts by the usual topological sheaf-theoretical arguments (namely, reduce to the affine case using general properties of sheaves and Proposition 1.1.6).

Proposition 1.5.2. *Let $f : X \longrightarrow Y$, $g : Y' \longrightarrow Y$ be morphisms and consider the cartesian diagram, where $X' = X \times_Y Y'$:*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y. \end{tikzcd}$$

Then:

$$\Omega_{X'/Y'} \cong (g')^*\Omega_{X/Y}.$$

Proposition 1.5.3. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphism of schemes. There is an exact sequence of $\mathcal{O}_X$-modules,*

$$f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0.$$

[3] Recall that $X \longrightarrow Y$ is a separated morphism, which is by-and-large the case of interest to us this semester, if and only if W is equal to $X \times_Y X$; that is, if and only if $\Delta(X)$ is closed in $X \times_Y X$.

Proposition 1.5.4. *Let $f : X \longrightarrow Y$ be a morphism of schemes and Z a closed subscheme of X defined by an ideal sheaf $\mathscr{I}$. There is an exact sequence of $\mathcal{O}_Z$-modules:*

$$\mathscr{I}/\mathscr{I}^2 \xrightarrow{\delta} \Omega_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega_{Z/Y} \longrightarrow 0.$$

2. Non-singular varieties

2.1. **A quick reminder concerning varieties over algebraically closed fields.** We give a quick and partial resumé of some results concerning varieties over an algebraically closed field k. One should really read through Hartshorne's Chapter I and Mumford's Chapter I to have anything similar to a descent introduction. We recall the kind of varieties we have:

(1) **Affine varieties.** These, by definition, are closed irreducible subsets V of $\mathbb{A}^n_k$ for some n. Namely, using that $k[x_1, \dots, x_n]$ is noetherian, they are defined by the vanishing of finitely many polynomials $f_1, \dots, f_m$ of $k[x_1, \dots, x_n]$ and are thought of as the sets of points $V = \{(v_1, \dots, v_n) \in k^n : f_j(v_1, \dots, v_n) = 0, j = 1, \dots, m\}$ (as opposed to considering the scheme $\mathrm{Spec}(k[x_1, \dots, x_n]/I)$, where $I = \langle f_1, \dots, f_m \rangle$). As in classical algebraic geometry one only deals with reduced schemes, the ideal I is assumed to be a radical ideal and, in fact, as varieties are irreducible, I is a prime ideal. Closed irreducible subvarieties of V correspond to prime ideals J containing I, namely to points of the scheme $\mathrm{Spec}(k[x_1, \dots, x_n]/I)$.

(2) **Quasi-affine varieties.** These are by definition open subsets of affine varieties. If $U \subset V$ is an open subset of a variety V then it follows that U is covered by open subsets of the sort V_f, where here we are using our scheme notation: $V_f = \mathrm{Spec}\ (k[x_1, \dots, x_n]/I)_f = \mathrm{Spec}\ k[x_1, \dots, x_n, y]/\langle I, yf - 1 \rangle$

(3) **Projective varieties.** They are, by definition, closed irreducible subsets of $\mathbb{P}^n_k$. We we have seen that they are determined by homogenous ideals $\mathfrak{a} \subseteq k[x_0, \dots, x_n]$.[4] From one of our main results in the previous semester it follows that any projective variety X is a proper scheme over k. Namely, the morphism $X \longrightarrow \mathrm{Spec}(k)$ is proper, and in particular for every variety V (affine, or projective) the morphism $X \times V \longrightarrow V$ is a closed map. In addition, X is separated.

(4) **Quasi projective varieties.** These are open subsets of projective varieties. Note that these are the most general. Any projective variety $V \subseteq \mathbb{P}^n$ defined by a homogenous ideal $\mathfrak{a}$ has a cover by affine open subvarieties $V = \cup_{i=0}^n V_i$, where V_i is defined by the elements of degree 0 in the localization of I in the variable x_i, namely, by $I[x_i^{-1}]^0$. Any of the previous varieties are in fact quasi-projective varieties. It follows that every quasi-projective variety is separated and has a basis consisting of affine varieties.

All those examples, viewed as schemes, are reduced, irreducible schemes of finite type over k. One can define an <u>abstract variety</u> to be a scheme which can be exhibited as a finite unions of such open subschemes. In general, abstract varieties are not quasi-projective anymore; namely, they are not necessarily isomorphic to subsets of any projective space. This flexibility is desired, in fact, and is

[4] Recall that we have seen that for any ring R closed subschemes of $\mathbb{P}^n_R$ can be defined by homogenous ideals $\mathfrak{a} \subseteq R[x_0, \dots, x_n]$. To get a bijection one should restrict attention to such ideals that satisfy $\mathfrak{a} : \langle x_0, \dots, x_n \rangle = \mathfrak{a}$. Namely, to ideals with the property that if $x_i f \in \mathfrak{a}$ for every $i = 0, \dots, n$ then $f \in \mathfrak{a}$. Note that this corrects a statement made last semester. It is indeed true that every homogenous ideal defines a closed subscheme and that every closed subscheme is coming from a homogenous ideal. But, as many different ideals may define the same closed subscheme, to get a <u>bijection</u> one needs to restrict to "saturated" ideals.

the reason for this more general definition. One can also define morphisms of abstract varieties in a direct manner. The key step is the definition for a morphism of affine varieties $V_1 \longrightarrow V_2$, where $V_i \subseteq \mathbb{A}^{n_i}$. Suppose that V_2 is defined by the polynomials $f_1, \dots, f_m$. Then,

$$\mathrm{Hom}(V_1, V_2) = \{(g_1, \dots, g_{n_2}) : g_i \in k[x_1, \dots, x_{n_1}], \forall \underline{a} \in V_1\ \forall j = 1, \dots, m, \\ f_j(g_1(\underline{a}), \dots, g_{n_2}(\underline{a})) = 0\}.$$

However, the category of abstract varieties over k is equivalent to the category of reduced, irreducible schemes of finite type over k with k-morphisms ([M], Theorem 2, page 88).

2.2. **Non-singular varieties: definition, differentials and the Jacobian criterion.** k is an algebraically closed field.

Definition 2.2.1. Let S be a scheme over $\mathrm{Spec}(k)$. We say that S is **regular at a point** x if the local ring $\mathcal{O}_{S,x}$ is a regular local ring, namely, if $\dim O_{S,x} = \dim_{\boldsymbol{k}(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2)$. We say that S is **regular** (or **non-singular**) if it is regular at each of its points.

Theorem 2.2.2. *Let S be an abstract variety.*

(1) *S is regular if the local ring of every closed point is regular.*
(2) *The local ring B of a closed point x is regular if and only if $\Omega_{B/k}$ is a free B-module of rank equal to $\dim B$. Moreover, S is regular if and only if $\Omega_{S/k}$ is a locally free module of rank equal to $\dim S$.*
(3) *Given a closed point $x \in S$, we can find a neighbourhood of x in S of the form $\mathrm{Spec}\ \ k[x_1, \dots, x_n]/\langle f_1, \dots, f_m\rangle$, where x corresponds to the point $\underline{0}$, i.e., to the ideal $\langle x_1, \dots, x_n\rangle$. Then S is non-singular at x, namely, the local ring of x is regular, if and only if*

$$n - \mathrm{rk}(\frac{\partial f_i}{\partial x_j}(\underline{0})_{i,j}) = \dim S.$$

Remark 2.2.3. Before the proof recall that if $V = \mathrm{Spec}(A)$ is an affine variety of k, $k(V) = \mathrm{Frac}(A)$ its field of functions then we have

$$\dim(V) = \dim(A) = \mathrm{tr.deg.}_k(k(V)) = \dim(A_{\mathfrak{p}}) + \dim(A/\mathfrak{p}),$$

for every prime ideal $\mathfrak{p}$ of A. Here $\dim(V)$ is the topological dimension and the dimension of a ring means its Krull's dimension.

Another tool required for the proof is Nakayama's lemma and a corollary of which.

Lemma 2.2.4 (Nakayama)**.** *Let $(A, \mathfrak{m})$ be a local ring and M a finitely generated A-module. If $\mathfrak{m}M = M$ then $M = 0$.*

Corollary 2.2.5. *Let M be an A module that is finitely generated. Suppose that $x_1, \dots, x_n$ are elements of M that generate the A-module $M/\mathfrak{m}M$. Then $x_1, \dots, x_n$ generate M.*

Proof. The first claim follows from a theorem stating that the localization of a regular local ring by a prime ideal is again a regular local ring. This theorem is actually not easy to prove and refer to [Mat] for the proof. However, given the theorem, since the local ring of any point of S can be obtained this way (reduce to an affine variety concerning the point in question), we are done.

For the second claim we first note that it follows from the nullstellensatz that if x is a closed point, B its local ring, then the composition $k \longrightarrow B \longrightarrow B/\mathfrak{m} = k$ is the identity. Now, suppose that $\Omega_{B/k}$ is free of rank equal to dim B. Then $\Omega_{B/k} \otimes_B k$ is a k-vector space of dimension equal to dim B but also of dimension equal to $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$ (Theorem 1.4.1). Therefore, B is a regular local ring.

Conversely, suppose that B is a regular local ring of dimension r. Thus, $\dim_k(\Omega_{B/k} \otimes k) = r$. On the other hand, if K is the fraction field of B then $\Omega_{B/k} \otimes_B K = \Omega_{K/k}$. As k is algebraically closed, hence perfect, K is separably generated over k and thus by Theorem 1.3.2 $\dim_K(\Omega_{K/k}) = \operatorname{tr.deg.}(K/k) = \dim B = r$ (where we have used results about dimension that we have already mentioned in the first term).

Now, it follows from our computations of differentials for quotients of polynomial rings that $\Omega_{B/k}$ is a finitely generated B-module. Thus, by Nakayama's lemma, there is a surjective homomorphism of B-modules

$$B^r \longrightarrow \Omega_{B/k}.$$

Let R be the kernel. We have an exact sequence

$$0 \longrightarrow R \longrightarrow B^r \longrightarrow \Omega_{B/k} \longrightarrow 0.$$

As K is flat over B, we get an exact sequence

$$0 \longrightarrow R \otimes_B K \longrightarrow K^r \longrightarrow \Omega_{K/k} \longrightarrow 0.$$

Our results about the dimension of $\Omega_{K/k}$ imply that $R \otimes_B K = \{0\}$ but $R \subset B^r$ and so is torsion free as B-module. This implies that $R = \{0\}$ and so that $\Omega_{B/k}$ is a free B-module.

Using localizations we conclude that if S is regular then $\Omega_{S/k}$ is locally free at every point of S - in the sense that if B is the local ring of that point then $\Omega_{B/k}$ is free B-module. As we have already used in the past, this implies that $\Omega_{S/k}$ is locally free - in the sense of existence of an open cover of S, $S = \cup S_i$ such that over each S_i, $\Omega_{S_i/k} = \Omega_{S/k}|_{S_i}$ is a free $\mathcal{O}_{S_i}$-module.

We now prove part (3). Let $\mathfrak{a} = (x_1, \dots, x_n)$. We have an isomorphism

$$\theta : \mathfrak{a}/\mathfrak{a}^2 \longrightarrow k^n, \qquad \theta(f) = \left(\frac{\partial f}{\partial x_1}(\underline{0}), \dots, \frac{\partial f}{\partial x_n}(\underline{0})\right).$$

Let $\mathfrak{b} = \langle f_1, \dots, f_m \rangle$. Note that $\mathfrak{a} \supseteq \mathfrak{b}$ and

$$n - \operatorname{rk}(\frac{\partial f_i}{\partial x_j}(\underline{0})_{i,j}) = n - \dim \theta(\mathfrak{b}).$$

Let $\mathfrak{m}$ be the maximal ideal of the point $\underline{0}$ of S. Then

$$\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{a}/(\mathfrak{b}+\mathfrak{a}^2) \cong (\mathfrak{a}/\mathfrak{a}^2)/((\mathfrak{b}+\mathfrak{a}^2)/\mathfrak{a}^2).$$

Therefore,

$$\begin{aligned}\dim_k(\mathfrak{m}/\mathfrak{m}^2) &= n - \dim_k((\mathfrak{b}+\mathfrak{a}^2)/\mathfrak{a}^2)\\ &= n - \dim_k \theta(\mathfrak{b})\\ &= n - \mathrm{rk}(\frac{\partial f_i}{\partial x_j}(\underline{0})_{i,j}).\end{aligned}$$

If B is the local ring of $\underline{0}$ on S then $\dim(B) = \dim(S)$ and B is regular if and only if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(B)$, i.e., if and only if

$$n - \mathrm{rk}(\frac{\partial f_i}{\partial x_j}(\underline{0})_{i,j}) = \dim(S).$$

□

We finally have a workable criterion for deciding non-singularity. Let's look at some examples:

Example 2.2.6. Consider the cuspidal curve C given by $y^2 - x^3$ in $\mathbb{A}^2$. Note that its field of functions is $k(x)[y]/(y^2 - x^3)$ which is of transcendence degree 1 over k. Thus C is really of dimension 1, justifying calling it a 'curve'. A point P on C is singular if and only if

$$(-3x^2(P), 2y(P)) = (0, 0).$$

This only happens (in any characteristic) if $P = (0, 0)$, which is indeed a point on the curve. Thus, over any field k there is a unique singular point, the point $P = (0, 0)$.

For a point $P = (a, b)$ on the curve, the tangent space is defined by the equation $-3a^2x + 2by = 0$. Note that this is a line at every point but $(0, 0)$, where it is a plane.

Example 2.2.7. Let C be the nodal curve $y^2 - (x^3 + x^2)$ in $\mathbb{A}^2$. In this case

$$(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (-3x^2 - 2x, 2y).$$

In characteristic different from 2, the singular points must satisfy $y = 0$ and thus that $x = 0, -1$ (else the point is not on the curve). But at $x = -1$, $-3x^2 - 2x = -1 \neq 0$. Thus, in characteristic different from 2, there is a unique singular point which is the point $(0, 0)$. At that point, the tangent space is equal to $\mathbb{A}^2$. In characteristic 2 the vanishing of $(-3x^2 - 2x, 2y)$ implies that $x = 0$, hence that $y = 0$ and so, again, the only singular point is $(0, 0)$.

Example 2.2.8. Consider the cubic curve $y^2z - x^3 - z^3$ in $\mathbb{P}^2$. Let us determine under which conditions it is non-singular. On the chart $z \neq 0$ it is given by $s^2 - t^3 - 1$ with partial derivatives $(2s, -3t^2)$. For characteristic different than $2, 3$, the vanishing of the partial derivatives implies $s = t = 0$, which is not a point on the curve. In contrast, in characteristic $2, 3$ there is a point on the curve s^2-t^3-1 for which both partial derivatives vanish. We thus assume that the characteristic is different from $2, 3$.

In the chart $y \neq 0$ we find the equation $s - t^3 - s^3$ with partial derivatives $(1 - 3s^2, -3t^2)$, the vanishing of which implies $t = 0$ and $s^2 = 1/3$, which is not a point on the curve. Simliarly, in the chart $x \neq 0$ we find the equation $y^2z - 1 - z^3$ with partial derivatives $(2yz, y^2 - 3z^2)$. We only need to consider the situation at the point $y = z = 0$, as all other points were already considered in other charts. But this point is not on the curve at all and we are done. That is, the curve is non-singular if and only if the characteristic of the base field is different from $2, 3$.

Exercise 2.2.9 (Exercises 6 - 9 in our count). Do the Exercises in Hartshorne, Chapter I, exercises 5.1, 5.2, 5.3, 5.5.

Theorem 2.2.10. *Let X be an abstract variety over k. Then the set of singular points X^{sing} of X is a closed proper subset of X.*

Proof. As X has a finite cover by open affine subsets, we may assume that $X \subseteq \mathbb{A}^n$ is affine, say $X = Z(I)$, where $I = (f_1, \dots, f_m)$ is a prime ideal. Suppose that $\dim(X) = r$. Then, by Theorem 2.2.2, the singular closed points of X are the points where

$$\text{rk}\left(\frac{\partial f_j}{\partial x_i}\right) < n - r.$$

Namely, it is the closed set defined by the vanishing of the determinant of all $(n - r) \times (n - r)$ sub matrices.

To show this is a proper closed set consider the generic point of X. There $\Omega_{K/k}$ is a free K-module of rank equal to tr. deg.$(K/k) = \dim(X)$ (as k is algebraically closed, hence perfect). Thus, $\Omega_{X/k}$ is free of rank equal to $\dim(X)$ in an open neighbourhood U of the generic point and all the points of U are non-singular by Theorem 2.2.2. □

2.3. **The tangent cone.** The tangent space informs us about where singularities occur, but it is not informative about the type of singularity. For example, at the singular points of both the cuspidal and the nodal curve the tangent space is two dimensional, but the behaviour of the curves at those points seems different. The tangent cone will allow us to capture this difference. The material in this section is given as a series of guided exercises. Let k be an algebraically closed field.

Exercise 2.3.1 (Exercise 10). Given a non-zero polynomial $f \in k[x_1, \dots, x_n]$ write f as a sum of its homogenous parts

$$f = f_r + \cdots + f_N,$$

where f_i is the homogenous part of f of weight i and $f_r \neq 0$. Define

$$f^* := f_r,$$

and define for an ideal I of $k[x_1, \ldots, x_n]$,

$$I^* = \langle f^* : f \in I \rangle.$$

Prove that I^* is a homogeneous ideal. Show by example that if $I = \langle f_1, \ldots, f_m \rangle$ then $I \supseteq \langle f_1^*, \ldots, f_m^* \rangle$, but they may not be equal. Show by example that I need not be a radical ideal.

Show, however, that if $I = \langle f \rangle$ is a principal ideal then $I^* = \langle f^* \rangle$. Calculate I^* for the cuspical and nodal curves.

Exercise 2.3.2 (Exercise 11). Let Y be an affine variety over k with coordinate ring $k[Y] = k[x_1, \ldots, x_n]/I$. Assume that $\underline{0} \in Y$. Define the tangent cone to Y at $\underline{0}$ as the scheme

$$C_{Y,\underline{0}} = \mathrm{Spec}(k[x_1, \ldots, x_n]/I^*).$$

Let us write $k[x_1, \ldots, x_n] = \oplus_{a=0}^{\infty} k[x_1, \ldots, x_n]_a$, the sum of the homogenous parts. Prove that if $I = \langle f_1, \ldots, f_m \rangle$ then $I^* \cap k[x_1, \ldots, x_n]_1 = \langle f_{1,1}, \ldots, f_{m,1} \rangle$. Deduce that the tangent space T to the tangent cone at $\underline{0}$ is equal to the tangent space $T_{Y,\underline{0}}$ of Y at $\underline{0}$ and that there is a natural closed immersion

$$C_{Y,\underline{0}} \hookrightarrow T_{Y,\underline{0}}.$$

Here are some examples:

Example 2.3.3. For the cuspidal curve $Y : y^2 = x^3$ we find that $C_{Y,\underline{0}} = \mathrm{Spec}(k[x, y]/(y^2))$. It embeds as a closed subscheme of $\mathbb{A}^2 = T_{Y,\underline{0}}$.

For the nodal curve $Y : y^2 = x^2(x + 1)$ we find that $C_{Y,\underline{0}} = \mathrm{Spec}(k[x, y]/((x + y)(x - y))$ which is a union of two lines crossing transversely at $\underline{0}$. It embeds as a reduced (but reduced) closed subscheme of $\mathbb{A}^2 = T_{Y,\underline{0}}$.

For the cone $Y : x^2 + y^2 = z^2$ in $\mathbb{A}^3$ we find that $C_{Y,\underline{0}} = Y$. The partial derivatives $(2x, 2y - 2z)$ vanish at $\underline{0}$. Thus, the tangent space at $\underline{0}$ is $\mathbb{A}^3$ in which the cone embeds as a reduced, irreducible closed subscheme.

Exercise 2.3.4 (Exercise 12). Give an example of a curve Y in $\mathbb{A}^3$, passing through $\underline{0}$, such that $T_{Y,\underline{0}} = \mathbb{A}^3$ and whose tangent space consists of lines whose linear span is $T_{Y,\underline{0}}$. In contrast give an example of a curve Y in $\mathbb{A}^3$, passing through $\underline{0}$, such that $T_{Y,\underline{0}} = \mathbb{A}^3$ and the reduced underlying scheme of $C_{Y,\underline{0}}$ is a single line.

Exercise 2.3.5 (Exercise 13). Let A be a local ring with maximal ideal $\mathfrak{m}$. Define the associated graded ring,

$$gr(A) = \oplus_{a=0}^{\infty} \mathfrak{m}^a/\mathfrak{m}^{a+1},$$

(where, by definition, $\mathfrak{m}^0 = A$). Let $k = A/\mathfrak{m}$ prove that $gr(A)$ is a graded k-algebra. Prove that if $x_1, \ldots, x_n$ generate $\mathfrak{m}/\mathfrak{m}^2$ then there is an isomorphism

$$gr(A) \cong k[x_1, \ldots, x_n]/I^*,$$

where I^* is some homogenous ideal of $k[x_1, \dots, x_n]$, where the isomorphism is as graded rings.

Suppose next that Y is an affine variety defined by an ideal I and that $\underline{0} \in Y$. Let $A = \mathcal{O}_{Y,\underline{0}}$, with maximal ideal $\mathfrak{m}A$, where $\mathfrak{m} = (x_1, \dots, x_n)/I$. Prove that

$$gr(A) \cong k[x_1, \dots, x_n]/I^*,$$

where I^* is the ideal generated by the leading homogenous terms of the elements of I. Conclude,

$$C_{Y,\underline{0}} \cong \mathrm{Spec}(gr(A)).$$

The preceding exercise has the important conclusion that the tangent cone is intrinsic; it can be defined as $\mathrm{Spec}(gr(A))$ and thus does not require introduction of coordinates. In particular, we have an elegant extension of the definition. For every abstract variety Y over k and a closed point $y \in Y$ define $C_{Y,y} \cong \mathrm{Spec}(gr(A))$, where $A = \mathcal{O}_{Y,y}$.

For the following lemma see [Eis], Exercise 13.8.

Lemma 2.3.6. *Let A be a noetherian local ring then* $\dim(A) = \dim(gr(A))$.

Corollary 2.3.7. *Let Y be an abstract variety over k and $y \in Y$ a closed point. Then*

$$\dim(C_{Y,y}) = \dim(Y).$$

Suppose now that Y is non-singular at y. Thus, $\dim T_{Y,y} = \dim(Y) = \dim C_{Y,y}$ and, as $C_{Y,y} \hookrightarrow T_{Y,y}$ we conclude that $C_{Y,y} = T_{Y,y}$ and $C_{Y,y}$ is reduced.

Conversely, suppose that $C_{Y,y} = T_{Y,y}$ then $\dim(Y) = \dim(C_{Y,y}) = \dim(T_{Y,y})$ and so Y is non-singular at y. To summarise:

Corollary 2.3.8. *Y is non-singular at y if and only if $C_{Y,y} = T_{Y,y}$.*

Exercise 2.3.9 (Exercise 14). The Cayley cubic is a singular surface given in $\mathbb{P}^3$ by the equation $\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0$, which we can write in polynomial form by multiplying by $x_0x_1x_2x_3$. Note that there is an action of S_4 on this surface.

Find the singular points of this surface. There are 4 of them. Show that any two singular points lie on a line lying on the surface. This gives 6 lines. Find the tangent cone at each singular point. Prove that there are at least 3 more lines on the Cayley cubic. One of them is given by the equations $x_0 + x_1 = x_2 + x_3 = 0$. In fact, these 9 lines are all the lines lying on the Cayley cubic, but this requires some work. Find the overall configuration of intersections between the 9 lines.

The Cayley cubic is the unique singular cubic in $\mathbb{P}^3$, up to isomorphism, with 4 ordinary double points and no other singular points (4 ordinary double points is in fact the maximal number of ordinary double points possible for a cubic surface).

3. Rational maps and birational equivalence

Once more k is an algebraically closed field. To simplify matters, in this section we shall only consider quasi-projective varieties over k. Since $\mathbb{P}^n_k$ is proper, it is separated and thus every quasi-projective variety is separated. Consequently, if $\varphi_i : X \longrightarrow Y, i = 1, 2$ are morphisms between quasi-projective varieties such that $\varphi_1|_U = \varphi_2|_U$ for some non-empty open subset U of X, then $\varphi_1 = \varphi_2$. More precisely, the set of points of X for which $\varphi_1 = \varphi_2$ is a closed set.

Definition 3.0.10. A rational map $X \overset{\varphi}{\dashrightarrow} Y$ is an equivalence class of pairs (U, α_U), where $U \subseteq X$ is an open non-empty set and $\alpha_U : U \longrightarrow Y$ is a morphism. The equivalence relation is defined by decreeing that $(U, \alpha_U) \sim (V, \alpha_V)$ if $\alpha_U|_{U\cap V} = \alpha_V|_{U\cap V}$. We call φ dominant if for some U (hence for any U) the closure of $\alpha_U(U) = Y$.

There is a well defined notion of composition of dominant rational maps $X \overset{\varphi}{\dashrightarrow} Y \overset{\psi}{\dashrightarrow} Z$ that allows for the following definition.

Definition 3.0.11. A birational map $X \overset{\varphi}{\dashrightarrow} Y$ is a dominant rational map such that there is a dominant rational map $Y \overset{\psi}{\dashrightarrow} X$ such that $\psi \circ \varphi = 1_X$ and $\varphi \circ \psi = 1_Y$ (equalities as rational maps). We say that X and Y are birationally equivalent and write $X \sim Y$ if there is a birational morphism $X \dashrightarrow Y$.

Now, given a variety X, we may associated to it its field of functions $k(X)$, which is, equivalently, (i) the local ring of the generic point of X, (ii) The fraction field of any open affine subvariety of X, (iii) the fraction field of the local ring of any point x of X. It is a field that is finitely generated over k.

Given a dominant rational map $X \overset{\varphi}{\dashrightarrow} Y$, the generic point of Y is necessarily the image of the generic point of X (to see that most easily, reduce to the case where X and Y are affine) and so we have an induced map $\varphi^* : k(Y) \longrightarrow k(X)$.

Lemma 3.0.12. *Let $\varphi_i :$ $X \overset{\varphi}{\dashrightarrow} Y$ be dominant rational maps such that $\varphi_1^* = \varphi_2^*$ then $\varphi_1 = \varphi_2$ as rational maps. Any ring homomorphism $k(Y) \longrightarrow k(X)$ comes from a dominant rational map $X \overset{\varphi}{\dashrightarrow} Y$.*

Proof. Let U be a non-empty open set on which both φ_i are defined. We note that the subset of U, $\{\varphi_1 = \varphi_2\}$, includes the generic point and is closed, hence equal to U. Thus, $\varphi_1 = \varphi_2$ are rational maps.

For the second claim, take any open affine $\mathrm{Spec}(A)$ in Y, where A is a finitely generated k-algebra: say $g_1, \dots, g_t$ are generators. Then $A \hookrightarrow k(Y) \hookrightarrow k(X) = \varinjlim \mathcal{O}_X(U)$, the limit taken over all non-empty open sets U of X. Under the composition, $g_i \mapsto h_i \in k(X)$ and so there exists an open, and thus open-affine, set U such that $h_i \in \mathcal{O}_X(U)$, $i = 1, \dots, t$. This gives us a morphism $U \longrightarrow \mathrm{Spec}(A)$, and thus a rational map $X \overset{\varphi}{\dashrightarrow} Y$ that induces φ. □

Theorem 3.0.13. *There is an anti-equivalence of categories between:*

(a) the category of quasi-projective varieties and dominant rational maps;

(b) finitely generated field extensions K/k with k-algebra homomorphisms.[5]

Proof. We have already constructed a function (a) $\mapsto$ (b) and we have shown that it is fully-faithful. The only thing remaining is to show essential surjectivity. Namely, that every finitely generated field extension K/k arises from some quasi-projective variety. Indeed, if K is generated by $g_1, \dots, g_n$ as a k-algebra the consider the map

$$k[x_1, \dots, x_n] \longrightarrow K, \qquad x_i \mapsto g_i.$$

The kernel is an ideal I, which is prime and defines an irreducible affine variety X. The function field of X is the field of fractions of $k[x_1, \dots, x_n]/I$ which is, on the one hand, contained in K but, on the other hand, contains generators for K/k, hence equal to K. □

Corollary 3.0.14. *The following are equivalent for two varieties X, Y over k.*

(1) $X \sim Y$;

(2) $k(X) \cong k(Y)$ as k-algebras;

(3) there exists non-empty isomorphic open sets $U \subseteq X$ and $V \subseteq Y$.

Proof. The equivalence of (i) and (ii) is the theorem. That (iii) implies (i) follows from the definitions directly. Suppose that (i) holds then there is an open non-empty subset $U_0 \subseteq X$ and a morphism $\varphi : U_0 \longrightarrow Y$ and a non-empty open subset $V_0 \subseteq Y$ and a morphism $\psi : V_0 \longrightarrow X$ such that whenever the compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are defined, they are the identity. Let $U = \varphi^{-1}(V_0)$ and $V = \varphi U$. The map $\varphi : U \longrightarrow V$ is surjective by definition and ψ is defined on V. Thus, $\varphi : U \longrightarrow V$ is an isomorphism. □

Corollary 3.0.15. *Every variety of dimension n is birational to a hypersurface in $\mathbb{P}^{n+1}$. In particular, any curve is birational to a plane curve and any surface is birational to a surface in $\mathbb{P}^3$.*

Proof. Let Y be a variety and $K = k(Y)$. As k is algebraically closed, K has a separating transcendence basis $\{x_1, \dots, x_n\}$ and $[K : k(x_1, \dots, x_n)] < \infty$ (being a finitely generated algebraic extension). By the primitive element theorem there is an irreducible polynomial $f(y) \in k(x_1, \dots, x_n)[y]$ such that $K = k(x_1, \dots, x_n)[y]/\langle f(y) \rangle$.

Suppose that $f = \sum a_i y^i, a_i \in k(x_1, \dots, x_n)$. We can multiply f by a non-zero polynomial $a \in k[x_1, \dots, x_n]$ (which is a unit of $k(x_1, \dots, x_n)$) without changing the ideal $\langle f(y) \rangle$ and thus we may as well assume each $a_i \in k[x_1, \dots, x_n]$. Therefore, K is the fraction field of $k[x_1, \dots, x_n, y]/\langle f(y) \rangle$.

[5] By that we mean that there are elements $g_1, \dots, g_t$ in K such that the minimal subfield of K containing k and all the g_i is K. This is different than saying the K is a k algebra of finite type.

This defines a hypersurface in $\mathbb{A}^{n+1}$ whose function field is K, and by taking the closure in $\mathbb{P}^{n+1}$ we get a projective hypersurface. $\square$

4. Curves

Although our main interest is to start the study of curves, we need some algebraic preliminaries. The local ring of a point on a curve will turn out to be a discrete valuation ring, which is a very powerful result. To understand such rings and how to prove that the local rings are such, we require some knowledge of valuation rings.

4.1. **Valuation rings.**

Definition 4.1.1. Let K be a field and G a linearly ordered abelian group[6]. Usually the group law on G is written additively. A valuation of K with values in G is a function

$$v : K^\times \longrightarrow G,$$

such that for all $x, y \in K^\times$:

(1) $v(xy) = v(x) + v(y)$;
(2) $v(x + y) \geq \min\{v(x), v(y)\}$.

Remark 4.1.2. It is easy, and useful, to check that $a, b \in G$ and $a \geq 0, b \geq 0$ implies $a + b \geq 0$ and $-a \leq 0$. Also, one deduces from the definition that $v(1) = 0 = v(-1)$.

Example 4.1.3. Here are the two standard examples:

(1) Let p be a prime number. Define a valuation on $\mathbb{Q}$, $v_p : \mathbb{Q}^\times \longrightarrow \mathbb{Z}$, by $v_p(x) = r$ if $x = p^r \cdot \frac{a}{b}$, where $p \nmid ab$. The valuation ring defined below - the elements of non-negative valuation - is $\mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at the multiplicative set $\mathbb{Z} - (p)$. That is, $\mathbb{Z}_{(p)} = \{\frac{a}{b} : a, b \in \mathbb{Z}, p \nmid b\}$ and its maximal idea is $p\mathbb{Z}_{(p)} = \{\frac{pa}{b} : a, b \in \mathbb{Z}, p \nmid b\}$.
(2) Let F be a field and let $f(t)$ be an irreducible monic polynomial of $F[t]$. Define $v_f : F(t)^\times \longrightarrow \mathbb{Z}$, by $v_f(g) = r$ if $g = f^r \cdot \frac{a}{b}$ where $a, b \in F[t], f \nmid ab$. The valuation ring is the localization $F[t]_{(f(t))}$ and its maximal ideal is $f(t) \cdot F[t]_{(f(t))}$.

Lemma 4.1.4. *Let v be a valuation on K. Let*

$$R = \{x \in K : x = 0 \text{ or } v(x) \geq 0\}.$$

Then R is a local ring with maximal ideal

$$\mathfrak{m} = \{x \in K : x = 0 \text{ or } v(x) > 0\}.$$

It has the property that for every $x \in K$ either $x \in R$ or $x^{-1} \in R$ (or both).

Conversely, let R be an integral domain with field of fractions K. Suppose that for all $x \in K$, either x or x^{-1} are elements of R (or both). Then R is a valuation ring.

[6] Thus G is a linearly ordered set and if $a \geq b$ then $a + c \geq b + c$ for all c.

Proof. The fact that R is a ring and $\mathfrak{m}$ is an ideal follow directly from the definition and the remark following it. If $x \in R - \mathfrak{m}$ then $v(x) = 0$ and thus $v(x^{-1}) = v(1) - v(x) = 0 - 0 = 0$. It follows that $x^{-1} \in R$. That is, every element of R not in $\mathfrak{m}$ is a unit. Thus, R is a local ring. Finally, as $v(x^{-1}) = v(1) - v(x) = -v(x)$ it follows that for every element x of K either x or x^{-1} are in R.

Conversely, let R be a domain with the stated properties. We build a valuation on K that would produce R as its valuation ring. Let

$$\Gamma = K^\times / R^\times,$$

which is an abelian group, written multiplicatively. For $\bar{a}, \bar{b} \in \Gamma$, say that

$$\bar{a} \geq \bar{b} \quad \text{ if } ab^{-1} \in R.$$

This is well defined, meaning independent of the chosen representatives for $\bar{a}, \bar{b}$. We note that:

(1) $\bar{a} \geq \bar{a}$;
(2) If $\bar{a} \geq \bar{b}$ and $\bar{b} \geq \bar{c}$ then ab^{-1} and bc^{-1} are in R and thus so is $ac^{-1} = (ab^{-1})(bc^{-1})$. That is, $\bar{a} \geq \bar{c}$.
(3) If $\bar{a} \geq \bar{b}$ and $\bar{b} \geq \bar{a}$ then both ab^{-1} and ba^{-1} are in R. Namely, $ab^{-1} \in R^\times$ and so $\bar{b} = \bar{b} \cdot \overline{ab^{-1}} = \bar{a}$ in Γ.
(4) For any $\bar{a}, \bar{b}$, either $\bar{a} \geq \bar{b}$ or $\bar{b} \geq \bar{a}$.
(5) Finally, if $\bar{a} \geq \bar{b}$ and $\bar{c}$ any element then $\bar{a}\bar{c} \geq \bar{b}\bar{c}$ as $ac(bc)^{-1} = ab^{-1} \in R$.

Thus, Γ is a linearly ordered abelian group. Define now a function

$$v : K^\times \longrightarrow \Gamma, \qquad v(a) = \bar{a}.$$

We claim that v is a valuation. Indeed, that $v(ab) = v(a)v(b)$ is clear. Suppose that $v(a) \geq v(b)$, namely, that $ab^{-1} \in R$. To show $v(a+b) \geq v(b)$ we need to show that $(a+b)b^{-1} \in R$. But $(a+b)b^{-1} = 1 + ab^{-1}$. Thus, we have shown that $v(a+b) \geq \min\{v(a), v(b)\}$. ☐

Definition 4.1.5. Let K be a field. Define an order on the set of local rings that are contained in K as follows. Say that $(A, \mathfrak{m}_A) \leq (B, \mathfrak{m}_B)$ if $A \subseteq B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$ (equivalently, $\mathfrak{m}_A \subseteq \mathfrak{m}_B$).

Theorem 4.1.6. *A valuation ring of K is a maximal element of the set of local rings of K. Any maximal element is in fact a valuation ring.*

Proof. In fact the proof is not that easy and we will have to sacrifice too much class time to explain it. We refer to [AM], Theorem 5.21. Here we only prove the easy part:

Let $(R, \mathfrak{m}_R)$ be a valuation ring and suppose that $(R, \mathfrak{m}_R) \leq (S, \mathfrak{m}_S)$. Let $x \in S$. If $x \in R$ we are done. Else, $x^{-1} \in R$, and in fact, $x^{-1} \in \mathfrak{m}_R$ (else, x^{-1} is a unit of R and so $x \in R$ leading us to the previous case). But then $x^{-1} \in \mathfrak{m}_S$, which implies that $x \notin S$. Contradiction. Thus, every $x \in S$ belongs to R and so $R = S$. ☐

Definition 4.1.7. A valuation $v : K^\times \longrightarrow G$ is called <u>discrete</u> if it is surjective and G is isomorphic to $\mathbb{Z}$ as an ordered abelian group.

One of the main results in this section is the following theorem:

Theorem 4.1.8. *Let A be a noetherian local domain of dimension* 1*. Let $\mathfrak{m}$ be its maximal ideal and $k = A/\mathfrak{m}$. The following are equivalent:*

(1) A is a discrete valuation ring.
(2) A is integrally closed in its fraction field K.
(3) $\mathfrak{m}$ is a principal ideal.
(4) A is a regular local ring, that is, $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$.

To prove the theorem we will use Nakayama's lemma; In addition, the following observations will be useful.

(1) Let $\mathfrak{a} \lhd A$ be a non zero ideal then $\mathfrak{a} \supseteq \mathfrak{m}^r$ for some r. Indeed, $\sqrt{\mathfrak{a}}$ is the intersection of prime ideals, by a general result about commutative rings we proved last term. But there is only one non-zero prime ideal, $\mathfrak{m}$. Thus, $\sqrt{\mathfrak{a}} = \mathfrak{m}$. As A is noetherian, $\mathfrak{m}$ is finitely generated and it follows that $\mathfrak{a} \supseteq \mathfrak{m}^r$ for a large enough integer r.
(2) For all $n \geq 0$, $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$. Else, we would have $\mathfrak{m}\mathfrak{m}^n = \mathfrak{m}^n$, which implies by Nakayama that $\mathfrak{m}^n = 0$. This contradicts the integral domain assumption (as $\mathfrak{m} \neq 0$ by the dimension 1 hypothesis).

Proof. (1) $\Rightarrow$ (2).
In fact, any valuation ring A is closed in its field of fractions K. If $x \in K$ is integral over A, then x satisfies $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$, for some $a_i \in A$. If $x \in A$ we are done. Otherwise $\frac{1}{x} \in A$ and we find that

$$x = -(a_{n_1} + \cdots + a_1(\frac{1}{x})^{n-2} + a_0(\frac{1}{x})^{n-1}) \in A.$$

(2) $\Rightarrow$ (3).
Let $a \in \mathfrak{m}, a \neq 0$. There exists some n such that $(a) \supseteq \mathfrak{m}^n$ but $(a) \not\supseteq \mathfrak{m}^{n-1}$. Choose then $b \in \mathfrak{m}^{n-1} - (a)$ and let $x = a/b$. We will show that $\mathfrak{m} = (x)$.

If $x^{-1} \in A$, i.e. $b/a \in A$, then $b \in (a)$. So that can't happen. Thus $x^{-1} \notin A$ and therefore x^{-1} is not integral over A. Now, if $x^{-1}\mathfrak{m} \subseteq \mathfrak{m}$ then, as $\mathfrak{m}$ is a finitely generated A-module (A is noetherian), if follows that x^{-1} is integral over A by a well-known criterion. Thus, $x^{-1}\mathfrak{m} \not\subseteq \mathfrak{m}$. On the other hand $x^{-1}\mathfrak{m} = \frac{b}{a}\mathfrak{m} \subseteq A$, because $b\mathfrak{m} \subseteq \mathfrak{m}^n \subseteq (a) = aA$. Thus, we must have $x^{-1}\mathfrak{m} = A$ and it follows that $\mathfrak{m} = xA$.

(3) $\Rightarrow$ (4)
If $\mathfrak{m} = (x)$ then $\mathfrak{m}^2 = (x^2)$ and $\mathfrak{m}/\mathfrak{m}^2 \cong k \cdot x$ is one dimensional (as $\mathfrak{m} \neq \mathfrak{m}^2$). Thus, A is a regular local ring.

(4) $\Rightarrow$ (1)

First, as A is a regular local ring of dimension 1 there is $x \in \mathfrak{m}$ such that $\mathfrak{m}/\mathfrak{m}^2 = k \cdot x$. By Nakayama's lemma, $\mathfrak{m} = (x)$. Thus, $\mathfrak{m}$ is principal. We shall now show that any non-zero ideal is a power of $\mathfrak{m}$, hence principal too.

Let $\mathfrak{a} \neq 0$ be a proper ideal of A. Then $\mathfrak{a} \subseteq \mathfrak{m}$ and $\mathfrak{a} \supseteq \mathfrak{m}^n$ for some n. It follows that there is an r such that $\mathfrak{a} \subseteq \mathfrak{m}^r = (x^r)$, but $\mathfrak{a} \not\subseteq \mathfrak{m}^{r+1}$ (else, we would also get that $\mathfrak{a} \subseteq \mathfrak{m}^r \subseteq \mathfrak{m}^n$ for r big enough, which would lead to $\mathfrak{m}^n = \mathfrak{m}^r$ for r sufficiently large, which would imply $\mathfrak{m}^n = \mathfrak{m}^{n+1}$). Thus, there is an element $a \in \mathfrak{a}$ such that $a = ux^r$ and $a \notin (x^{r+1})$; that is, $u \notin (x)$. But that implies that u is a unit of A, which gives $x^r \in \mathfrak{a}$. Consequently, $\mathfrak{a} = (x^r)$.

Now, given $a \in A$ we have $(a) = (x^r)$ for some $r \geq 0$. Define then

$$v(a) = r.$$

It is easy to check that this gives a valuation on K, when defining $v(a/b) = v(a) - v(b)$, making A into a discrete valuation ring. $\square$

We would often be interested in the situation where $k \subset K$ is a subfield and we consider valuations that are trivial on k. The local ring of such a valuation is a valuation ring and so a maximal local ring of K relative to the order we defined above. On the other hand, given a local ring of K that contains k, note that it is the same to say that it is maximal among all local rings of K, or maximal among all local rings of K that contain k. And in this case it is a valuation ring such that the valuation on k is trivial. We therefore conclude,

Corollary 4.1.9. *Let $k \subset K$ be a subfield. There is a bijection between valuation of K whose valuation is trivial on k (considered up to equivalence of the value group of the valuation) and local rings of K that are maximal, relative to domination, among all local rings of K that contain k.*

4.1.1. *Geometric applications.*

Corollary 4.1.10. *Let X be a non-singular quasi-projective curve over an algebraically closed field k. Let $x_0 \in X$ be a closed point. Then $\mathcal{O}_{X,x_0}$ is a discrete valuation ring with residue field k. Denote the valuation v_{x_0}.*

Given $f \in k(X)$, the function field of X (which is the local ring of the generic point of X), call $v_{x_0}(f)$ the order of vanishing of f at x_0.

Proof. As X is non-singular, all the local rings are regular. The local ring of a closed point is a regular local ring of dimension 1, hence a discrete valuation ring. $\square$

More generally, given a variety V over an algebraically closed field k and a point x on it, $\mathcal{O}_{V,x}$ can be calculated using any affine chart. Thus, we may assume $V = \operatorname{Spec} R$ is an affine variety and x corresponds to a prime ideal $\mathfrak{p}$. We have already used in the past that

$$\dim(V) = \dim(A/\mathfrak{p}) + \operatorname{ht}(\mathfrak{p}).$$

Note also that $\mathrm{ht}(\mathfrak{p}) = \dim(A_\mathfrak{p})$. Thus, if $\mathfrak{p}$ is of height 1, namely, corresponds to a closed irreducible subvariety $Z \subset V$ of codimension 1, and if $\mathfrak{p}$ is a regular point, which would be the case if, say, V is non-singular, then $A_\mathfrak{p}$ is a discrete valuation ring. The associated valuation v satisfies that $v(f)$ is the order of vanishing of f along Z.

Example 4.1.11. Take $X = \mathbb{A}^1_k$ with coordinate t and $x_0 = 0 = \langle t \rangle$. Let f be a polynomial and write $f = a_r t^r + a_{r+1}t^{r+1} \cdots + a_{r+n}t^{r+n}$ where $a_r \neq 0$. Then $v_0(f) = r$. This valuation extends to $K = k(t)$.

The same applies to any other point $\alpha \in \mathbb{A}^1_k$, which corresponds to the maximal ideal $\langle t - \alpha \rangle$. Similarly, for $f = a_r(t-\alpha)^r + \cdots + a_{r+n}(t-\alpha)^{r+n}$ with $a_r \neq 0$ we have $v_\alpha(f) = r$.

There is yet another valuation on K, v_∞. It measures the order of vanishing at infinity. Namely, if we view K as the function field of $\mathbb{P}^1_k$ this is precisely the valuation corresponding to the point at infinity. This valuation is characterized by

$$v_\infty(t^{-1}) = 1.$$

This valuation satisfies the relation

$$v_\infty(f(t)) = v_0(f(1/t)).$$

Thus, for example $v_\infty(\frac{1}{t+1}) = v_0(\frac{t}{t+1}) = v_0(t) - v_0(1+t) = 1$.

An interesting point to note is that by considering all the valuations on K we were able to discover that there is a missing point, i.e. the point ∞.

Example 4.1.12. Consider the local ring A at $\underline{0}$ on the curve $y^2 = x^3$. This ring is Noetherian of dimension one, being the localization of a Noetherian ring of dimension one in a maximal ideal.

The Jacobian criterion shows that the point $\underline{0}$ is singular. Thus, the conditions specified in Theorem 4.1.8 fail. It is interesting to see how. Firstly, the maximal ideal is $\mathfrak{m} = \langle x, y \rangle \pmod{I}$, where $I = \langle y^2 - x^3 \rangle$. Thus, we find that

$$\mathfrak{m}/\mathfrak{m}^2 \cong k \cdot x \oplus k \cdot y,$$

has dimension 2. Thus, the ring is not regular and, at the same time, we see that $\mathfrak{m}$ cannot be a principal ideal. We also note that, as expected, A is not integrally closed. Let $t = yx^{-1}$. Then $t^2 = y^2x^{-2} = x$ and thus $t^2 - x = 0$. Thus, t is integral over A, but t doesn't belong to A.

Finally, A is not a dvr. Although we can define a function v on A by $v(y) = 3$ and $v(x) = 2$ and it extends naturally to a function on A (and its field of fractions), A is not the valuation ring of v; The valuation ring contains at least $k[x, y, t]/(y^2 - x^3)$, which is isomorphic to $k[t]$. As $k[t]$ is integrally closed it follows that the valuation ring is precisely $k[x, y, t]/(y^2 - x^3)$.

Exercise 4.1.13. Do the exercises [H] II.4.5 (a), (b).

Exercise 4.1.14. This exercise is taken from [AM] Exercises 28 and 32, page 72.

Let Γ be a totally ordered abelian group. A subgroup Δ of Γ is called isolated in Γ if, whenever $0 \leq \beta \leq \alpha$ and $\alpha \in \Delta$ then $\beta \in \Delta$. (Perhaps a better name would have been convex.)

(1) Let A be a valuation ring with fraction field K and value group Γ (in particular, $v : K^* \to \Gamma$ is surjective). Let $\mathfrak{p}$ be a prime ideal of A. Show that $v(A - \mathfrak{p})$ is the set of non-negative elements of an isolated subgroup Δ of Γ. Show further that the mapping so defined of $\mathrm{Spec}(A)$ into the set of isolated subgroups of Γ is bijective. (One defines the rank of the valuation as the length n of a maximal chain of isolated subgroups $\Delta_0 \subsetneq \cdots \subsetneq \Delta_n$. Note that this is therefore just the Krull dimension of A).
(2) Deduce from this correspondence that the set of prime ideals of A is totally ordered.
(3) If $\mathfrak{p}$ is a prime ideal, prove that $A/\mathfrak{p}$ and $A_{\mathfrak{p}}$ are valuation rings as well. What are the value groups for these valuations?

Exercise 4.1.15. (Example of a valuation ring of rank 2). Consider the abelian group $\mathbb{Z}^2$ with the lexicographic order: $(a, b) < (a', b')$ if either $a < a'$ or $a = a'$ and $b < b'$. Show that this is a linearly ordered abelian group. Find its isolated subgroups.

We now proceed to finding a field with a valuation in this group. Let K be the field of formal power series in two variables and complex coefficients satisfying the following restrictions: every element of K is a power series $\sum_{r \geq a}(x^r \sum_{s \geq b(r)} c_{r,s} y^s)$, where a is an integer and $b(r)$ is an integer depending on r.

(1) Show that K is a field.
(2) Given an element of K as above, define its valuation as the minimal (r, s) for which $c_{r,s} \neq 0$.
(3) Find the valuation ring and its prime ideals.

4.2. **Curves: The idea, the goal and some consequences.** Let Y be an affine curve with coordinate ring $A(Y)$. We are familiar with the correspondence

$$\{\text{closed points } y \in Y\} \quad \longleftrightarrow \quad \{\text{maximal ideals of } A(Y)\},$$

given by $y \mapsto \mathfrak{m}_y$. If Y is projective, we have a difficulty constructing a similar correspondence. Of course, we could use the homogenous coordinate ring of Y and its maximal ideals, but unlike in the affine case, the homogenous coordinate ring depends on the embedding of Y in a projective space and typically two different embeddings do not yield isomorphic rings. Nevertheless, we note that there is another approach that works in the affine case. There is an injective map

$$Y \hookrightarrow \{\text{local rings of } k(Y)\}, \qquad y \mapsto \mathcal{O}_{Y,y}.$$

This map is indeed injective, because for different points we localize $A(Y)$ at different maximal ideals to obtain the local rings. Note that we can recover the maximal ideals by $\mathfrak{m}_{Y,y} \cap A(Y)$. This approach generalizes well to the case where Y is quasi-projective as the local ring of a point is an intrinsic notion that doesn't depend on the projective embedding. Thus, for Y quasi-projective, we have a function

$$Y \longrightarrow \{\text{local rings of } k(Y)\}, \qquad y \mapsto \mathcal{O}_{Y,y}.$$

Lemma 4.2.1. *This map is injective.*

Proof. Let $P \neq Q$ be distinct points of Y and say $Y \subseteq \mathbb{P}^n$. We claim that there exists a function f that is zero at P and non-zero at Q. Let $g = 1/f$. Then $g \in \mathcal{O}_{Y,Q}$ and $g \notin \mathcal{O}_{Y,P}$. This shows that the local rings are distinct. Now to the construction of f: Think about P and Q as vectors in $\mathbb{A}^{n+1}$. As linear functionals separate vectors, there is a linear functional ϕ that vanishes at P and not at Q and there is a linear functional ψ that vanishes at neither. We can write $\phi = a_0x_0 + \cdots + a_nx_n$, $\psi = b_0x_0 + \cdots + b_nx_n$ and we let $f = \frac{\phi}{\psi} = \frac{a_0x_0+\cdots+a_nx_n}{b_0x_0+\cdots+b_nx_n}$. $\square$

Let K/k be a **function field of dimension** 1; that is K/k is a finitely generated field extension such that the transcendence degree of K over k is 1. These are precisely the fields arising the field of rational functions of a curve Y over k. Call a dvr $R \subset K$ a dvr of K/k if the valuation gives value 0 to $k^\times$ and the fraction field of R is K (which is, in fact, always the case, but we shall not need to use that).

Consider now a particular case where Y is a non-singular curve over k. In that case, we have an injection

$$Y \hookrightarrow \{\text{dvr's of } k(Y)/k\}, \qquad y \mapsto \mathcal{O}_{Y,y}.$$

One of the main points of this chapter is that if Y is projective, this is a bijection. This suggest that, in some sense, a projective non-singular curve Y should be thought of as the collection of dvr's of $k(Y)/K$. After developing a language allowing us to make sense of this idea, we will be able to prove one of the main results of this chapter. Namely:

The following three categories are equivalent:

(1) *Projective non-singular curves and dominant morphisms.*
(2) *Quasi-projective curves with dominant rational maps.*
(3) *Function fields K/k of dimension 1 and k-algebra homomorphisms.*

We remark that the equivalence of (2) and (3) - to be precise, an anti-equivalence - is already known to us from our discussion of rational morphisms and birational equivalence. There is also a canonical function from (1) to (2): A projective curve is in particular a quasi-projective curve and a dominant morphism is an example of a dominant rational map.

Let us now illustrate what the equivalence means. For example, it implies the following: Given any quasi-projective curve Y, there is a smooth projective curve X, such that $X \sim Y$, equivalently $k(X) = k(Y)$. Moreover, X is unique up to isomorphism. Further, given a dominant rational map $Y_1 \overset{f}{\dashrightarrow} Y_2$, and smooth projective curves $X_i \sim Y_i$, there exists a unique morphism $\phi : X_1 \longrightarrow X_2$

such that the following diagram commutes

$$\begin{tikzcd} X_1 \arrow[r, "\phi"] \arrow[d, dashed] & X_2 \arrow[d, dashed] \\ Y_1 \arrow[r, dashed, "f"] & Y_2 \end{tikzcd}$$

In particular, if Y_1, Y_2 are themselves smooth projective curves and $Y_1 \overset{f}{\dashrightarrow} Y_2$ is a rational map then f extends uniquely to a morphism $Y_1 \longrightarrow Y_2$.

For example, we can use these statements to conclude that if Y is a smooth projective curve and $Y_0 \subseteq Y$ is an affine curve then any automorphism $Y_0 \longrightarrow Y_0$ extends uniquely to an automorphism $Y \longrightarrow Y$.

Before we start, we remark that any dominant morphism $X \longrightarrow Y$, where $X \longrightarrow \mathrm{Spec}(k)$ is proper, is surjective (as the image is closed and dense). In particular, <u>any dominant morphism between projective curves (singular or not) is surjective.</u>

Remark 4.2.2. One may ask whether the approach taken here for curves can be generalized to higher dimensional varieties. Let us explain one aspect of the complications arising. It follows from our main results that every function field of dimension 1 is the function field of a unique non-singular projective curve C and, moreover, given any projective curve C' with function field $k(C)$ there is a dominant birational morphism $C \to C'$. Already for surfaces the situation is very different.

Let S be a non-singular projective surface and $k(S)$ its function field. There can certainly be other non-isomorphic non-singular projective surfaces S' with the same function field such that, although, S and S' are birational, there is no dominant morphism $S \to S'$ or $S' \to S$. Nevertheless, at least for surfaces we have a very good understanding how all such surfaces are related to each other. Yet, fix such a surface S. By taking an irreducible curve C on S we get a discrete valuation associated with it (and recall that scheme-theoretically we can think about C as a point, viz. the generic point of C). But there are infinitely many discrete valuations rings of $k(S)/k$ that do not arise this way. For example, take S and blow-it up at a point s_0. This is a process we shall discuss later, but for now let us allow that it provides a nonsingular projective surface S' and a birational morphism $\pi : S' \to S$ that is an isomorphism outside $\pi^{-1}(s_0)$ and $\pi^{-1}(s_0)$ is an irreducible curve C. The valuation ring of C is a discrete valuation ring of $k(S') = k(S)$, but "is not visible" on S itself. The same process can now be repeated for S' and any point s_0' on it, whether lying on C or not, and so on! It is not clear, even if we look at the set of all valuation rings of $k(S)/k$, which model it should prefer. No model will capture, for example, all its discrete valuation rings.

4.3. **Abstract non-singular curves.** Let K/k be a function field of dimension 1; as always, k is an algebraically closed field. Let C_K be the set of dvr's of K/k. The following lemma will be used repeatedly.

Lemma 4.3.1. *The following holds:*

(1) *Let $x \in K$. Then $\{R \in C_K : x \notin R\}$ is a finite set.*
(2) *Let $y \in K$, $y \neq 0$. Then $\{R \in C_K : y \in \mathfrak{m}_R\}$ is a finite set.*

Proof. The case $x = 0$ is trivial. Suppose that $x \neq 0$ and put $y = 1/x$. If $x \notin R$ then $y \in \mathfrak{m}_R$, and vice-versa. Therefore, it is enough to prove the second statement of the lemma.

If $y \in k^\times$, then $y \in R^\times$ for any $R \in C_K$, so the set $\{R \in C_K : y \in \mathfrak{m}_R\}$ is empty. Let then $y \in K - k$. Since k is algebraically closed, y is transcendental over k. If $y \in \mathfrak{m}_R$ then, in particular,

$$k[y] \subset R \subset K.$$

Let B be the integral closure of $k[y]$ in K. Since $K/k(y)$ is algebraic and finitely generated, B is a finitely generated k-algebra that is a finite $k[y]$-module, by the "finiteness of integral closure" theorem in algebra (see [H] Theorem 3.9A). Moreover, the quotient field of B is K (every element of K is integral over some localization of $k[y]$ - look at the minimal polynomial of that element over $k(y)$). As all the local rings of B at prime ideals are noetherian, dimension 1 and integrally closed they are regular local rings. Thus, B corresponds to a smooth affine curve Y with $A(Y) = B$ and $k(Y) = K$. (Namely, $Y = \operatorname{Spec} B$ and $A(Y)$, the affine coordinate ring of Y, is just another notation for $\Gamma(Y, \mathcal{O}_Y)$.)

Using the notation $N_K(-)$ to denote normal closure in K, we note that $k[y] \subset R$ implies that $B = N_K(k[y]) \subseteq N_K(R) = R$, because R is integrally closed and its fraction field is K. We conclude that $\underline{B \subset R \text{ for any dvr } R \text{ of } K/k \text{ such that } y \in \mathfrak{m}_R.}$ Let

$$\mathfrak{n}_R = \mathfrak{m}_R \cap B.$$

The ideal $\mathfrak{n}_R$ is a prime ideal of B, which is a Dedekind ring, namely a noetherian integrally closed domain of dimension 1. Therefore $\mathfrak{n}_R$ is a maximal ideal, corresponding to some point $P \in Y$. We have

$$\mathcal{O}_{Y,P} = B_{\mathfrak{n}_R} \subseteq R.$$

Note that by Theorem 4.1.8, $B_{\mathfrak{n}_R}$ is a dvr of K/k too. Since dvr are maximal local rings relative to domination and R dominates $B_{\mathfrak{n}_R}$, we have that $B_{\mathfrak{n}_R} = R$. Thus, $y \in \mathfrak{m}_R$ implies that $y \in \mathfrak{m}_P$. That is, y vanishes at P. But $y \neq 0$ so it vanishes at finitely many points. □

The proof gives also the following conclusion.

Corollary 4.3.2. *Any dvr of K/k is the local ring of some point on a smooth affine curve Y with $k(Y) = K$. In particular, $R/\mathfrak{m}_R = k$.*

To define abstract non-singular curves - or AC for short - we consider the following space (in fact, a special case of AC).

- The points of the space are C_K. It is a space with infinitely many points as revealed by the proof of Lemma 4.3.1.

- The topology is the co-finite topology.
- We define a sheaf of functions $\mathcal{O}$: for $U \subseteq C_K$ we let

$$\mathcal{O}(U) = \cap_{R \in U} R.$$

We note that by said lemma, the function field of this space, namely $\varinjlim_{U \neq \emptyset} \mathcal{O}(U)$, is just K. Also, the local ring of a point R, namely $\varinjlim_{U \text{ s.t. } R \in U} \mathcal{O}(U) = R$ (use Lemma 4.3.1).

Now, every element of $f \in \mathcal{O}(U)$ defines a "real" function $f : U \longrightarrow k$ by the formula

$$f(R) = f \pmod{\mathfrak{m}_R} \in R/\mathfrak{m}_R = k$$

(we shall refer to f itself, temporarily, as "abstract" function). The "real" function f determines the "abstract" function f; indeed, if f and g define the same "real" function, then $f - g \in \mathfrak{m}_R$ for all $R \in U$, which is an infinite set. By Lemma 4.3.1, $f - g = 0$. We can therefore easily identify "abstract" functions with "real" functions.

Definition 4.3.3. An abstract non-singular curve, or AC for short, is a non-empty open set U of C_K (for some function field K/k of dimension 1) with the induced topology and sheaf of regular functions. It is thus a locally ringed space.

We will see shortly that we may think about AC as a curve but until we have established that, if we want to consider morphisms between varieties and AC's, we have to enlarge the category of varieties by including also all AC for any function field K/k of dimension 1.

If V_1, V_2 are objects are objects of this enlarged category then a morphism $f : V_1 \longrightarrow V_2$ is a continuous function such that for all $U \subseteq V_2$ open and a "real" function $g : U \longrightarrow k$, the "real" function $g \circ f : f^{-1}(U) \longrightarrow k$ is a regular function. That means that if V_1 is curve, this "real" function arises from an "abstract" function in the manner discussed above. We get a category this way that contains the category of varieties.

4.4. **Curves and abstract curves.**

Proposition 4.4.1. *Every non-singular quasi-projective curve Y is isomorphic to some AC.*

Proof. Let $K = k(Y)$ and $U \subseteq C_K$ be the set

$$U = \{\mathcal{O}_{Y,P} : P \in Y\}.$$

We shall show below that U is open. Suppose that for the time being. Then U is an AC. Define

$$\varphi : Y \longrightarrow U, \qquad P \mapsto \mathcal{O}_{Y,P}.$$

As we have noted before, this is a bijection. Let $V \subseteq Y$ be open, then

$$\mathcal{O}(V) = \cap_{P \in Y} \mathcal{O}_{Y,P}.$$

(This just expresses the fact that being regular is a local property.) It follows that

$$\mathcal{O}(V) = \mathcal{O}(\varphi(V))$$

and therefore that φ is an isomorphism.

Now, to show U is open it is enough to show that $C_K - U$ is finite and so it is enough to show that U contains a non-empty open set. We may therefore assume that Y is affine. In this case the proof of Lemma 4.3.1 shows that

$$U = \{\text{dvr } R \text{ of } K/k : R \supset A(Y)\}.$$

Let $x_1, \dots, x_n$ be generators of $A(Y)$ as a k-algebra. Then,

$$U = \{R \in C_K : x_i \in R, i = 1, \dots, n\} = \cap_{i=1}^{n}\{R \in C_K : x_i \in R\}.$$

By Lemma 4.3.1, each set $\{R \in C_K : x_i \in R\}$ is co-finite and so U is co-finite too. $\square$

The following proposition, besides being important for the development of our final result, is of great interest because it provides an effective procedure for extending morphisms.

Proposition 4.4.2. *Let X be an AC, $P \in X$ and Y a projective variety. Any morphism*

$$\varphi : X - \{P\} \longrightarrow Y$$

extends uniquely to a morphism $X \longrightarrow Y$.

Proof. Suppose that $Y \subseteq \mathbb{P}^n$, then the morphism $\varphi : X - \{P\} \longrightarrow Y$ induces a morphism $\varphi : X - \{P\} \longrightarrow \mathbb{P}^n$. Suppose that this morphism can be extended to $\varphi : X \longrightarrow \mathbb{P}^n$, then $\varphi^{-1}(Y)$ is closed and contains $X - \{P\}$, hence equal to X (closed sets, except for X itself, are finite). Therefore the morphism $\varphi : X \longrightarrow \mathbb{P}^n$ necessarily factors through Y and gives us an extension $\varphi : X \longrightarrow Y$. Note that this extension is unique, because two morphisms agreeing on an open dense set, $X - \{P\}$ in our case, are equal everywhere.

Thus, we may consider the problem of extending a morphism $\varphi : X - \{P\} \longrightarrow \mathbb{P}^n$ to a morphism $\varphi : X \longrightarrow \mathbb{P}^n$.

Let $U \subseteq \mathbb{P}^n$ be the open set whose points are $\{a \in \mathbb{P}^n : a_i \neq 0, i = 0, \dots, n\}$. We may assume that $\varphi(X - \{P\}) \cap U \neq \emptyset$. Indeed, if not, then $\varphi(x - \{P\}) \subset \cup_{i=0}^{n} Z(x_i)$. As $X - \{P\}$ is irreducible (proper finite sets are finite, after all) so is $\varphi(x - \{P\})$ and thus, there is an i such that $\varphi(x - \{P\}) \subseteq Z(x_i) \cong \mathbb{P}^{n-1}$. Thus, making use that the case of $n = 0$ it trivial and arguing by induction, we may assume this doesn't happen and so that $\varphi(X - \{P\}) \cap U \neq \emptyset$.

Let $f_{ij} = \varphi^*(x_i/x_j)$. This is a regular function of $\varphi^{-1}(U)$, which is a non-empty open set. Thus, $f_{ij} \in K$. Let us denote the valuation on P (P, recall, is a dvr) by v, and let $r_i = v(f_{i0})$. Then

$$v(f_{ij}) = v(f_{i0}/f_{j0}) = r_i - r_j.$$

Choose an a such that

$$v(f_{a0}) = \min\{v(f_{00}), v(f_{10}), \dots, v(f_{n0})\}.$$

Then,

$$v(f_{ia}) = r_i - r_a \geq 0, \forall i.$$

That is, $f_{0a}, \ldots, f_{na} \in P$. Extend φ by defining

$$\varphi(P) = (f_{0a}(P), \ldots, f_{na}(P)).$$

This is well-defined as all the functions f_{ia} are regular at P and not all vanish at P: $f_{aa} \equiv 1$. We need to show φ is a morphism. To begin with, to show that φ pulls-back regular functions to regular functions, it is enough to deal with an arbitrarily small open set containing $\varphi(P)$.

Note that $\varphi(P) \in V := \{x; x_a \neq 0\}$ and that is an affine open subset of $\mathbb{P}^n$ with affine coordinate ring $k[x_0/x_a, \ldots, x_n/x_a]$. As $\varphi^*(x_i/x_a) = f_{ia}$ is regular at P, and regularity at any other point of $\varphi^{-1}(V)$ is already known, φ^* takes regular functions on V to regular functions on $\varphi^{-1}(V)$. Given a $V_1 \subseteq V$ open and $g \in \mathcal{O}(V)$ it follows easily, by writing g locally as a fraction of regular functions on V, that $\varphi^*(g)$ is regular on $\varphi^{-1}(V_1)$.

Finally, to show φ is continuous, we need to show that the pre-image of a closed set is closed. Note that $\varphi(X - \{P\})$, as well as its closure, is connected and irreducible. It is thus either a point or a curve. The first case is trivial. In the second case, as "most" closed sets are finite, a quick examination of the situation reveals that the only problem may occur when there is closed set $Z \supseteq \varphi(X - \{P\})$ such that $\varphi(P) \notin Z$. But then, if we define a function g by $g(Z) \equiv 0, g(P) = 1$, then g is a regular function on $\varphi(X)$. Thus, $\varphi^*(g)$ is regular on X; but this is a function that is zero on $X - \{P\}$ and 1 at P, and that's a contradiction. $\square$

Here are some remarks concerning Proposition 4.4.2:

(1) The proposition applies to the case where X is a quasi-projective smooth curve, because we know such are isomorphic to abstract curves.

(2) The proposition may fail if Y is not projective. Let $X = \mathbb{P}^1, P = (1:0), Y = \mathbb{A}^1$ and

$$\varphi : \mathbb{P}^1 - \{(1:0)\} \longrightarrow \mathbb{A}^1, \qquad (x : y) \mapsto x/y.$$

Then φ doesn't extend to $\mathbb{P}^1$ as any morphism from a projective variety to an affine variety is constant.

(3) The proposition may fail when $\dim(X) > 1$. For example, let $X = \mathbb{A}^2, P = (0,0)$, Y some projective closure of $\mathrm{Bl}_P(\mathbb{A}^2)$. Let

$$\varphi : \mathbb{A}^2 - \{(0,0)\} \longrightarrow Y, \qquad (a_1, a_2) \mapsto (a_1, a_2; a_1 : a_2) \in \mathrm{Bl}_P(\mathbb{A}^2) \subset Y.$$

This φ is an isomorphism from $\mathbb{A}^2 - \{(0,0)\}$ to the open set $\mathrm{Bl}_P(\mathbb{A}^2) - \mathrm{E}$ (E - the special fibre) that cannot be extended to X.

(4) The proof of Proposition 4.4.2 makes use of the fact that we can write a function into a projective space in many ways. If locally around P,

$$Q \mapsto (f_0(Q) : \cdots : f_n(Q)),$$

then we may say that this is also the map (up to rational equivalence)

$$Q \mapsto \left(\left(\frac{f_0}{f_a}\right)(Q) : \cdots : \overset{a}{1} : \cdots : \left(\frac{f_n}{f_a}\right)(P)\right).$$

The index a was chosen so that to make this expression well defined at P, and hence locally around P. This proof is very useful for explicit computations.

For instance, consider the nonsingular elliptic curve $X = Z(g(x, y, z)) \subseteq \mathbb{P}^2_{x,y,z}$ given by $y^2z - (x^3 + z^3) = 0$. We define a morphism

$$\varphi : X - \{(0:1:0)\} \longrightarrow \mathbb{P}^1, \qquad (x : y : z) \mapsto (x : z).$$

We know now that this morphism can always be extended to X and how to do it. The proof tells us to consider the two functions $1, \frac{z}{x}$ at the local ring of the point $P = (0 : 1 : 0)$. If $\frac{z}{x}$ is regular at P then extend φ by $\varphi(P) = (1 : \frac{z}{x}(P))$. Else, necessarily $\frac{x}{z}$ is regular at P and extend φ by $\varphi(P) = (\frac{x}{z}(P) : 1)$. Note that if both $\frac{x}{z}$ and $\frac{z}{x}$ are regular, then indeed $(1 : \frac{z}{x}(P)) = (\frac{x}{z}(P) : 1)$.

We have phrased this that way just to indicate that the method is completely general. In our case, passing to the affine chart $y = 1$, we have the relation $z = z^3 + x^3$. Let v denote the valuation of the point P. We clearly have $v(z) > 0$ and $v(x) > 0$. If $v(x) \geq v(z)$ then $v(z) = v(z^3 + x^3) \geq v(z^3) = 3v(z)$ which is a contradiction. Thus, $v(x) < v(z)$ and so $v(z) = 3v(x)$. (Another way to perform these calculations is to note that $x^3 = z(1 - z^2)$ and $1 - z^2$ is a unit at P. Thus, $3v(x) = v(z)$.) As the maximal ideal at P is clearly generated by x and z there is no choice but $v(x) = 1, v(z) = 3$; in particular, $v(z/x) = 2$. Thus, the function z/x is regular at P and the extension we are looking for is $P \mapsto (1 : 0)$.

Theorem 4.4.3. *The abstract curve C_K is isomorphic to a projective non-singular curve Y.*

Proof. Every point $R \in C_K$ has an open neighbourhood U^R isomorphic to an affine non-singular curve Y^R. As $C_K - U^R$ is finite, we can write

$$C_K = \cup_{i=1}^m U_i, \qquad U_i \cong Y_i^\circ \subseteq Y_i,$$

where Y_i° is a non-singular curve and Y_i its closure in some projective space $\mathbb{P}^{n_i}$. By Proposition 4.4.2, the morphism $\varphi_i : U_i \longrightarrow Y_i$ extends to a morphism

$$\varphi_i : C_K \longrightarrow Y_i.$$

Consider the product $\prod_{i=1}^m Y_i$ which is a closed irreducible subset of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$ and so a projective variety. We have a morphism

$$\varphi = (\varphi_1, \dots, \varphi_m) : C_K \longrightarrow \prod_{i=1}^m Y_i.$$

Let Y be the closure of $\varphi(C_K)$. Note that $Y \subset \prod_{i=1}^m Y_i$ and so the projection maps $p_i : \prod_{i=1}^m Y_i \longrightarrow Y_i$ are defined on Y. $\varphi(C_K)$ is dense in Y and has dimension 1 as its projection onto Y_i is dominant (for any i). Y is thus a curve and $k(Y) \subseteq K$. Our goal is to show that the morphism $\varphi : C_K \to Y$ is an isomorphism.

Let $R \in C_K$. Then $R \in U_i$ for some i. We have the following commutative diagram

$$\begin{tikzcd} C_K \arrow[r, "\varphi"] & Y \arrow[d, "p_i"] \\ U_i \arrow[u, hook] \arrow[r, "\varphi_i", "\text{open immers.}"'] & Y_i \end{tikzcd}$$

All the morphisms are dominant and we conclude that

$$R = \mathcal{O}_{U_i,R} \cong \mathcal{O}_{Y_i,\varphi_i(R)} \subseteq \mathcal{O}_{Y,\varphi(R)} \subseteq \mathcal{O}_{C_K,R} = R.$$

As φ_i is an isomorphism, we get equalities through out, and so $\mathcal{O}_{Y,\varphi(R)} \cong R$. This implies that $k(Y) = K$ and that φ is injective (recall, that if $x, y \in Z$, a quasi-projective variety, then $\mathcal{O}_{Z,x} = \mathcal{O}_{Z,y} \Leftrightarrow x = y$).

The morphism φ is also surjective: Let $P \in Y$. We claim that there exists a dvr R of K/k such that $R \supseteq \mathcal{O}_{Y,P}$. Indeed, as Y is a curve, there exists an open set $U \subseteq Y$ that is affine and such that $P \in U$. Let $\tilde{U}$ be its normalization in $k(Y)$. We have a finite birational morphism $f : \tilde{U} \longrightarrow U$. Let $\tilde{P}$ be a point of $\tilde{U}$ such that $f(\tilde{P}) = P$. Then, $\mathcal{O}_{Y,P} = \mathcal{O}_{U,P} \subseteq \mathcal{O}_{\tilde{U},\tilde{P}}$, which is a dvr R since $\tilde{U}$ is a non-singular curve. We obtain then a point $R \in C_K$ such that

$$\mathcal{O}_{Y,\varphi(R)} = R \supseteq \mathcal{O}_{Y,P}.$$

We claim that this implies $P = \varphi(R)$; that is, if x, y are points on a curve Z and $\mathcal{O}_{Z,x} \subseteq \mathcal{O}_{Z,y}$ then $x = y$. For that, it is enough to show that if $x \neq z$ then there exists a function on Z that vanishes at x and not at y to obtain a contradiction. Repeat the argument of Lemma 4.2.1.

At this point we know that $\varphi : C_K \longrightarrow Y$ is a bijective morphism. But, we can cover C_K and Y by open sets U_i and $\varphi(U_i)$ respectively, on which φ restricts to an isomorphism, because the composition $U_i \longrightarrow C_K \xrightarrow{\varphi} Y \xrightarrow{p_i} Y_i$ is the open immersion φ_i. Namely, the inverse of $\varphi|_{U_i}$ is the morphism $\varphi_i^{-1} \circ p_i$. Thus, φ is an isomorphism. □

Corollary 4.4.4. *Any AC is isomorphic to some non-singular quasi-projective curve.*

Proof. We have $U \subseteq C_K \cong Y$, where Y is a non-singular projective curve and so U can be identified with an open subset of Y and therefore is a quasi-projective non-singular curve. □

Corollary 4.4.5. *Every non-singular quasi-projective curve Y° is isomorphic to an open subset of a non-singular projective curve Y.*

Proof. Indeed, by Proposition 4.4.1, Y° is an AC. Use the previous corollary. □

Corollary 4.4.6. *Every curve Y' is birationally equivalent to a non-singular projective curve Y, $Y' \sim Y$.*

Proof. Indeed $Y' \sim Y^\circ$, where Y° is the non-singular locus of Y'. Use the previous corollary. ☐

4.5. **An equivalence of categories.**

Theorem 4.5.1. *The following categories are equivalent:*

1. *Projective non-singular curves and dominant morphisms.*
2. *Quasi-projective curves with dominant rational maps.*
3. *Function fields K/k of dimension 1 and k-algebra homomorphisms.*

Proof. We already know the equivalence of (2) and (3) as a special case of Corollary 3.0.14; the rest follows from our results above. ☐

4.5.1. *Normalization.* Let X be a quasi-projective curve. Then there is a quasi-projective non-singular curve and a finite birational morphism $\tilde{X} \longrightarrow X$. To show that, it is enough to consider the case where X is projective, the general case follows by restricting to a subset. In that case, let $\tilde{X}$ be the non-singular projective model of X. The inclusion $K(X) \subseteq K(\tilde{X})$ produce a rational morphism $\tilde{X} \longrightarrow X$, which, by Proposition 4.4.2, extends to a morphism $f : \tilde{X} \longrightarrow X$. As the inclusion of function fields is actually an equality, this morphism is birational. Further, $\tilde{X}$ is normal, being non-singular. The morphism f is in fact surjective. Given a point $x \in X$, choose a dvr R of K/k such that $\mathcal{O}_{X,x} \subseteq R$. R corresponds to a point $t \in \tilde{X} \cong C_K$. Consider $f(t)$. If $f(t) \neq x$, we can find a rational function g on X that vanishes at $f(t)$ and is invertible at x. Then $f^*(g)$ vanishes at t and so is in the maximal ideal of R, but is a unit in $\mathcal{O}_{X,x}$. A contradiction.

We can cover X by open affine subsets U such that $f^{-1}(U)$ is affine. But then $f : f^{-1}(U) \longrightarrow U$ produces an injection of rings $A(U) \subseteq A(f^{-1}(U))$. Passing to integral closure in $K(X)$ we get $B(U) \subseteq A(f^{-1}(U))$, as $A(f^{-1}(U))$ is integrally closed, being a ring of regular functions of a non-singular affine curve, and where we have let $B(U)$ be the integral closure of $A(U)$. We claim that $B(U)$ is equal to $A(f^{-1}(U))$.

Let t be a point of $A(f^{-1}(U))$ and R the corresponding local ring. Let R_1 be the local ring of $f(t)$. Then $R_1 \subseteq R$ and both are dvr of K/k. We saw that this implies $R = R_1$ (see the proof of Lemma 4.3.1). This, in turn implies that the map $f : f^{-1}(U) \longrightarrow U$ is injective because if $f(t_1) = f(t_2)$ then t_1, t_2 have the same local ring and so are equal. Thus, the inclusion $B(U) \subseteq A(f^{-1}(U))$ is surjective too. We have equality. That means that locally $\tilde{X}$ is the normalization of X and so $\tilde{X}$ is the normalization of X.

4.6. **Morphisms between curves.**

Lemma 4.6.1. *A proper curve C, namely a curve for which the structural morphism $C \longrightarrow \mathrm{Spec}(k)$ is proper, is a projective curve and vice-versa.*

Proof. If C is projective then it is a closed subscheme of $\mathbb{P}^n_k$. As $\mathbb{P}^n_k \longrightarrow \mathrm{Spec}(k)$ is a proper morphism, so is $C \longrightarrow \mathrm{Spec}(k)$.

Suppose now that $C \longrightarrow \mathrm{Spec}(k)$ is proper. C is an open subset of a projective curve $\bar{C}$ with the same field of functions $k(C)$. The tautological map,

$$C \times \bar{C} \longrightarrow \bar{C},$$

is closed as $C \longrightarrow \mathrm{Spec}(k)$ is proper. The subset $C \times C \subseteq C \times \bar{C}$ is closed as it is equal to $\bar{C} \times \bar{C} \cap C \times \bar{C}$. But its image under the morphism above is C, which is closed in $\bar{C}$ if and only if $C = \bar{C}$. $\square$

Proposition 4.6.2. *Let X, Y be non-singular projective curves over k and*

$$f : X \longrightarrow Y,$$

a dominant morphism. Then f is a finite flat morphism.

Proof. We have already noted that f is surjective. Let $x \in X$ and $y = f(x)$. Then $\mathcal{O}_{Y,y}$ is a discrete valuation ring and $\mathcal{O}_{X,x}$ is an $\mathcal{O}_{Y,y}$-module. It is thus flat over $\mathcal{O}_{Y,y}$ if and only if it is torsion-free. But this is clear as both are contained in the same integral domain $k(X)$ (via $\mathcal{O}_{Y,y} \hookrightarrow k(Y) \hookrightarrow k(X)$).

Now, as $X \longrightarrow Y \longrightarrow \mathrm{Spec}(k)$ is a factorization of the proper morphism $X \longrightarrow \mathrm{Spec}(k)$ one concludes that $X \longrightarrow Y$ is proper as well (cf. [H], Corollary II.4.8). In addition, it is clear that every point of Y has finitely many pre-images so $X \longrightarrow Y$ is quasi-finite too. At this point we could have used a general result that a proper quasi-finite morphism of locally noetherian schemes $X \longrightarrow Y$ is finite (theorem of Chevalley). This is proved usually as a rather easy consequence of Zariski's main theorem that we did not have the occasion to discuss. But this is an over-kill in our case and we can provide a different argument. We still need to rely on some results in algebra we hadn't proved but conceptually it is an easier proof.

Note that $k(X)$ is a finite algebraic extension of $k(Y)$. Let $V = \mathrm{Spec}(B)$ be any open affine subset of Y and let A be the integral closure of B in $k(X)$. It is a finite B-module (cf. [H] I.3.9.A). On the other hand, the local rings of $\mathrm{Spec}(A)$ are dvr's, so we may think about $U = \mathrm{Spec}(A)$ as an open subset of $C_{k(X)} = X$. Furthermore, $U = f^{-1}(V)$. Indeed, from the perspective of abstract curves, U is the set of valuation rings of $k(X)/k$ that contain A. But as each valuation ring is integrally closed we can also say that U are the valuation rings that contain B. However, the set of valuation rings of a field K that contain a subring B is exactly the integral closure of B in K (cf. [H] II Theorem 4.11 A).

That way, we get a finite cover of $Y = \cup V_i$ such that $X = \cup f^{-1}(V_i)$ and each $f^{-1}(V_i) \longrightarrow V_i$ is a finite morphism. It follows that f is a finite morphism. $\square$

Corollary 4.6.3. *Let $f : X \to Y$ be a dominant morphism of projective non-singular curves. Let $Y^0 \subset Y$ be an affine curve then $f^{-1}(Y^0)$ is an affine curve. In particular, let $f : X \to \mathbb{P}^1$ be a rational non-constant function then $f^{-1}(\mathbb{A}^1)$ is an affine curve.*

Proof. In fact, the proof of the Proposition shows that if $\mathrm{Spec}(A) \subseteq Y$ then $f^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(B)$, where B is the integral closure of A in $k(X)$. We may also argue as follows: a finite morphism has the property that the pre-image of any affine subset is affine. This gives the first claim. For the second, first note that if $f \in k(X)$ is non-constant then f certainly gives a rational map $X \dashrightarrow \mathbb{P}^1$, which is in fact a morphism to $\mathbb{A}^1$ when we remove the poles of f from X. Thus, f extends to a morphism $X \to \mathbb{P}^1$ and we apply then previous claim. $\square$

Remark 4.6.4. Later we will be able to apply that to prove that any quasi-projective curve that is not projective is affine. In particular, removing one point (or more) from a projective curve yields an affine curve. What we are missing is the fact that if X is a projective curve, $x_0 \in X$, then there is a non-constant function $f \in k(X)$ whose only poles are at the point x_0. This is a consequence of the Riemann-Roch theorem.

4.6.1. *Degree.* Let $f : X \longrightarrow Y$ be a dominant morphism between projective non-singular curves. Define the **degree** of f as

$$\deg(f) := [k(X) : k(Y)].$$

This is a finite number, as $k(X)$ and $k(Y)$ are both of transcendence degree 1 over k and finitely generated over k. As f is finite-flat, we know from the first semester that for every point $y \in Y$ we have

$$\deg(f) = \sum_{f(x)=y} \deg_{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}).$$

If we write $f^{-1}(y)$ scheme-theoretically as a $\boldsymbol{k}(y)$-algebra say A_y. Then

$$\deg(f) = \dim_{\boldsymbol{k}(y)}(A_y).$$

Furthermore, it follows from Theorem 4.5.1 that $\deg(f) = 1$ if and only if f is an isomorphism.

Proposition 4.6.5. *Let X be a non-singular projective curve and $f \in k(X)$ a non-singular rational function. Define the divisor of f as*

$$(f) := \sum_{P \in X(k)} v_P(f)[P].$$

The degree of (f), namely $\sum_P v_P(f)$, is 0.

Proof. We know that the degree of f over 0 and over ∞ are equal. Namely, we know that $\dim_{\boldsymbol{k}(y)}(A_y)$ is constant as y ranges over $\mathbb{P}^1$. Now the fibre A_y is the spectrum of an Artinian k-algebra and factors according to the maximal ideals that correspond to the actual points in X that map to y. Namely, $A_y \cong \coprod_{f(x)=y} \mathcal{O}_{X,x}/\mathfrak{m}_{\mathbb{P}^1,y}\mathcal{O}_{X,x}$. The only thing remaining is to relate the length of $\mathcal{O}_{X,x}/\mathfrak{m}_{\mathbb{P}^1,y}\mathcal{O}_{X,x}$ to $v_x(f)$. However, $\mathfrak{m}_{\mathbb{P}^1,y}\mathcal{O}_{X,x}$ is a principal ideal of $\mathcal{O}_{X,x}$ and it is determined by the valuation of a generator of it. A generator of it is simply $f^*(\pi)$, where π is a local uniformizer at the point y. The confusing thing is actually the identification of $k(\mathbb{P}^1)$ with $k(t)$. The identification is so that t is a uniformizer at 0. Then f <u>is</u> $f^*(t)$. At ∞ it is $u = 1/t$ that is a uniformizer and we have $f^*(u) = f^*(1/t) = 1/f^*(t)$, as f^* is a ring homomorphism. Thus, at every point x mapping to ∞ we have $v_x(f^*(u)) = -v_x(f)$. $\square$

Example 4.6.6. Hyperelliptic curves. We ask to classify all the diagrams

$$f : X \longrightarrow \mathbb{P}^1,$$

where X is a non-singular projective curve and f is a surjective morphism of degree 2. Such a curve X is called **hyperelliptic**. Note that what we are doing is classifying all pairs (X, f) up to isomorphism, which is not the same as classifying all X up to isomorphism. Using Theorem 4.5.1, this is the same as classifying all quadratic extensions

$$k(t) = K(\mathbb{P}^1) \subset K.$$

The discussion brakes now naturally into two cases.

(1) The characteristic of k is not 2. In this case, Kummer's theory applies. Such extensions correspond canonically to non-trivial elements of $k(t)^\times/k(t)^{\times 2}$. To a polynomial $g(t)$, which is not a square, one associate the curve $y^2 = g(t)$, equivalently, the function field $k(t)[y]/(y^2 - g(t))$.
(2) The characteristic of k is 2. Here one uses Artin-Schreier theory. Such extensions correspond canonically to nontrivial elements in $k(t)/S$, where $S = \{f^2-f : f \in k(t)\}$. To a polynomial $g(t)$ one associates the curve $y^2-y = g(t)$, that is, the function field $k(t)[y]/(y^2-y-g(t))$.

Finally we list some exercises about curves.

Exercise 4.6.7. Show that the affine curves given by $y = x^2$ and $xy = 1$ are birational but not isomorphic.

Exercise 4.6.8. Show further, that for every irreducible quadratic polynomial $f(x, y) \in k[x, y]$ the conic section defined by $f(x, y) = 0$ in $\mathbb{A}^2$ is isomorphic to precisely one of the curves above and give a criterion to determine which. (This is [H] Ex. I 1.1, which is much easier to do once we have all the theory we have developed!)

Exercise 4.6.9. Show that the group $\mathrm{PGL}_2(k) := \mathrm{GL}_2(k)/k^*$ acts faithfully as automorphisms of $\mathbb{P}^1_k$ via the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} t := \frac{at + b}{ct + d}$$

(Möbius transformations), where we have identified the function field of $\mathbb{P}^1_k$ with that of $\mathbb{A}^1_k = \mathrm{Spec}\, k[t]$. Show further that any automorphism of $\mathbb{P}^1_k$ arises this way. That is

$$\mathrm{Aut}_k(\mathbb{P}^1_k) = \mathrm{PGL}_2(k).$$

(It is also true that $\mathrm{Aut}(\mathbb{P}^n_k) = \mathrm{PGL}_{n+1}(k)$.)

Exercise 4.6.10. Let $P_1, \dots, P_a$ be distinct closed points of $\mathbb{A}^1_k$ and $Q_1, \dots, Q_b$ another distinct set of distinct points of $\mathbb{A}^1_k$. Prove that if $\mathbb{A}^1 - \{P_1, \dots, P_a\} \cong \mathbb{A}^1 - \{Q_1, \dots, Q_b\}$ then $a = b$. Show that the converse may fail - what is a minimal counter-example?

Exercise 4.6.11. Consider the projective curve $C_d : x^d + y^d + z^d = 0$ in $\mathbb{P}^2_k$ (the Fermat curve). Assume k has characteristic 0 (to simplify the calculations). Show that the rational map $(x : y : z) \mapsto (x : y)$ defines a dominant morphism $C_d \longrightarrow \mathbb{P}^1_k$. Calculate the degree of this map. Determine all points $(x : y)$ in which the closed points of the fibre have cardinality smaller than the degree and determine precisely the cardinality of the fibre at those points.

The curve C has a large group of automorphisms. Which of those automorphisms commutes with the morphism $C_d \longrightarrow \mathbb{P}^1$? Is the field extension $k(C_d) \supseteq k(\mathbb{P}^1)$ Galois?

Exercise 4.6.12. Show that the non-singular curve associated to the cuspidal curve $y^2 = x^3$, as well as to the nodal curve $y^2 = x^2(x + 1)$, is $\mathbb{P}^1$. In both cases provide a surjective birational morphism from $\mathbb{P}^1$ to the closure of the curve in $\mathbb{P}^2$, namely to $y^2z - x^3 = 0$ and $y^2z - x^2(x + 1) = 0$.

To continue further our study of curves we need the machinery of <u>cohomology</u>, which is likewise instrumental in studying varieties in general. Therefore, we will now take a rather long break from curves and varieties and go back to sheaf theory.

5. Cohomology of sheaves

Our purpose in this section is to give a coherent description of cohomology of sheaves on schemes that is also efficient. We will be citing many facts; in fact, almost anything that we think we can cite without making the exposition too vague, we shall. Our main interest is to have a working knowledge of cohomology. Experience shows that to a large extent this allows one to consider many of the statements in cohomology as a black box. Nonetheless, we don't want to be that reckless and, for some applications, one really needs to understand how the constructions go. So some compromise is called for.

Most of the exposition is based on Hartshorne's book. Although, when we later on return to cohomology to introduce spectral sequences, we shall be following Mumford's "second red book".

5.1. **Abelian categories.** Let $\mathbf{C}$ be a category. There is a notion of $\mathbf{C}$ being an abelian category. One assumes that $\mathbf{C}$ has a zero object and that finite products and co-products exist in $\mathbf{C}$: $\mathrm{Hom}_{\mathbf{C}}(A, B)$ is an abelian group for any two objects of $\mathbf{C}$, composition is additive relative to this structure and morphisms have kernels and cokernels. It is this last requirement that makes this notion a bit of a headache. There is no a-priori notion of a sub object and morphisms are not functions, a priori. Thus, one needs to invent a notion of kernel. If $f : A \longrightarrow B$ is a morphism, its kernel (which is assume to exist) is an object K together with a morphism $i : K \longrightarrow A$ such that the composition $K \longrightarrow A \longrightarrow B$ is the zero map and, moreover, given any morphism $i' : K' \longrightarrow A$ such that the composition $K' \longrightarrow A \longrightarrow B$ is zero, the morphism i' factors through a unique morphism $K' \longrightarrow K$. In short, this is hardly elegant. The same goes for a definition of co-kernel. Thus, we will assume that we are in situation where the kernel is always a sub object (and that this statement makes sense) and the cokernels are quotient objects. The prototypical example of an abelian category is the category of modules over a given ring R (not necessarily commutative). The embedding theorem says that every (small) abelian category is equivalent to a full sub-category of the category of modules over a ring R. This justifies our pedestrian point of view that allows for kernels and cokernels to always be viewed as sub and quotient objects, respectively.

The basic example of the category of modules over a ring R allows one to show that the category of sheaves of abelian groups on a topological space X, and the category of quasi-coherent sheaves on a locally ringed space (e.g., a scheme), are abelian categories. We have discussed in the past the correct definition of a kernel of a morphism of schemes (the straightforward construction) and co-kernel (sheafify the pre-sheaf of co-kernels).

Definition 5.1.1. A complex $A^\bullet$ in an abelian category $\mathbf{C}$ is a sequence of objects and morphisms of $\mathbf{C}$,

$$\cdots \longrightarrow A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \longrightarrow \cdots,$$

such that $d^{i+1} \circ d^i = 0$ for all i. If $A^\bullet$ is a complex, we define

$$H^i(A^\bullet) = \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1})$$

and call it the i-th cohomology group of $A^\bullet$.

Definition 5.1.2. A morphism of complexes $f^\bullet : A^\bullet \longrightarrow B^\bullet$ is a collection of morphisms $\{f^i : A^i \longrightarrow B^i\}$ such that

$$f^{i+1} \circ d^i = d^{i+1} \circ f^i.$$

$f^\bullet$ induces a homomorphism

$$H^i(A^\bullet) \longrightarrow H^i(B^\bullet), \qquad \forall i \in \mathbb{Z}.$$

The following diagram makes the above definition and induced maps on cohomology easier to check:

$$\begin{tikzcd}
\cdots \arrow[r] & A^i \arrow[r, "d^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r, "d^{i+1}"] \arrow[d, "f^{i+1}"] & A^{i+2} \arrow[r] \arrow[d, "f^{i+2}"] & \cdots \\
\cdots \arrow[r] & B^i \arrow[r, "d^i"] & B^{i+1} \arrow[r, "d^{i+1}"] & B^{i+2} \arrow[r] & \cdots
\end{tikzcd}$$

Suppose that

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

is an exact sequence of complexes; that is, for every i the sequence $0 \longrightarrow A^i \longrightarrow B^i \longrightarrow C^i \longrightarrow 0$ is exact. Then there is a long exact sequence of cohomology groups

$$\begin{tikzcd}
 & & \cdots \arrow[dll] \\
H^i(A^\bullet) \arrow[r] & H^i(B^\bullet) \arrow[r] & H^i(C^\bullet) \arrow[dll, "\delta^i"] \\
H^{i+1}(A^\bullet) \arrow[r] & H^{i+1}(B^\bullet) \arrow[r] & H^{i+1}(C^\bullet) \arrow[dll, "\delta^{i+1}"] \\
\cdots & &
\end{tikzcd}$$

The maps δ^i appearing here are derived from the snake lemma applied to the diagrams

$$\begin{tikzcd}
0 \arrow[r] & A^i \arrow[r] \arrow[d, "d^i"] & B^i \arrow[r] \arrow[d, "d^i"] & C^i \arrow[r] \arrow[d, "d^i"] & 0 \\
0 \arrow[r] & A^{i+1} \arrow[r] & B^{i+1} \arrow[r] & C^{i+1} \arrow[r] & 0
\end{tikzcd}$$

5.2. **Resolutions.** An object I in an abelian category $\mathbf{C}$ is injective if the functor $\mathrm{Hom}_{\mathbf{C}}(-, I)$, which is always left exact, is exact. Namely, given a short exact sequence in $\mathbf{C}$, $0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$, we have an exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{C}}(A_3, I) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(A_2, I) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(A_1, I) \longrightarrow 0.$$

Otherwise said, I is injective if and only if

$$A_1 \hookrightarrow A_2 \quad \Rightarrow \quad \mathrm{Hom}_{\mathbf{C}}(A_2, I) \twoheadrightarrow \mathrm{Hom}_{\mathbf{C}}(A_1, I).$$

Definition 5.2.1. Let A be an object in $\mathbf{C}$. An injective resolution of A is an exact sequence in $\mathbf{C}$,

$$0 \longrightarrow A \xrightarrow{\xi} I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots,$$

where each I^j is an injective object. If every object of $\mathbf{C}$ has an injective resolution we say that $\mathbf{C}$ has enough injectives.

The main results are the following. Let R be a commutative ring. The category of R-modules has enough injectives. In fact, one first proves it for the case $R = \mathbb{Z}$, and the general case is deduced from that.

An abelian group I is injective if and only if it is divisible. Namely, for every positive integer m and any element $a \in I$ there is an element $b \in I$ such that $mb = a$. That injective implies divisible is an easy exercise. The other direction is not. Given the relationship between "injective" and "divisible", one deduces that (arbitrary) direct sums and quotients of divisible modules are divisible. So, for example, $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}^r \oplus (\mathbb{Q}/\mathbb{Z})^s$ are injective $\mathbb{Z}$-modules.

Exercise 5.2.2. Prove that every abelian abelian group can be embedded in an injective $\mathbb{Z}$-module. (Hint: start by writing A as a quotient of a free abelian group). Deduce that the category of abelian groups has enough injectives.

The general case now follows (and that is at a level of an exercise) from the case of abelian groups as follows. Let M be an R-module and let D be a divisible abelian group in which M embeds as an abelian group. There is a natural map $M \to \mathrm{Hom}_{\mathbb{Z}}(R, D)$ taking $m \in M$ to the homomorphism ϕ_m, where $\phi_m(x) = xm$. One then proves that $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ has a natural R-module structure, that the map $M \to \mathrm{Hom}_{\mathbb{Z}}(R, D)$ is an injective R-module homomorphism and that $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is an injective R-module. This last point is proven via canonical identifications between Hom's (namely, $\mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(R, D)) = \mathrm{Hom}_{\mathbb{Z}}(A, D))$ by reducing to the case of abelian groups.

As one can imagine, we can boot-strap those results to prove that the category of sheaves of abelian groups on a topological space has enough injectives, or that the category of $\mathcal{O}_X$-modules on a scheme X, has enough injectives ([H] III 2.2).

5.3. **Right-derived functors and cohomology.** We finally arrive to one of the key definitions in cohomology of sheaves. Let $\mathbf{C}, \mathbf{D}$, be abelian categories and let

$$F : \mathbf{C} \longrightarrow \mathbf{D}$$

be a covariant, left-exact, additive functor. Assume that $\mathbf{C}$ has enough injectives. The right-derived functors $R^iF, i \geq 0$ of F,

$$R^iF : \mathbf{C} \longrightarrow \mathbf{D},$$

are defined as follows. Given an object $A \in \mathbf{C}$, we choose an injective resolution

$$0 \longrightarrow A \xrightarrow{\xi} I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots,$$

and let $I^\bullet$ be the complex

$$I^\bullet : \quad 0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots,$$

where I^0 is considered as the degree 0 term of this complex. We apply F and get another complex, this time in $\mathbf{D}$,

$$F(I^\bullet) : \quad 0 \longrightarrow F(I^0) \xrightarrow{d^0} F(I^1) \xrightarrow{d^1} F(I^2) \xrightarrow{d^2} \cdots$$

We let

$$R^iF(A) := H^i(F(I^\bullet)) := \frac{\mathrm{Ker}(d^i : F(I^i) \longrightarrow F(I^{i+1}))}{\mathrm{Im}(d^{i-1} : F(I^{i-1}) \longrightarrow F(I^i))}.$$

Here are the key properties of this construction:

(1) The definition of $R^iF(A)$ is independent of the resolution up to a natural isomorphism. (This requires the notion of homotopic resolutions; any two injective resolutions are homotopic.)
(2) For every i, $R^iF : \mathbf{C} \longrightarrow \mathbf{D}$ is a covariant additive functor and, in particular, $R^iF(A \oplus B) = R^iF(A) \oplus R^iF(B)$.
(3) There is a natural isomorphism $R^0F \cong F$.
(4) A short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ induces a long exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \xrightarrow{\delta^0} R^1F(A) \longrightarrow R^1F(B) \longrightarrow \dots$$
$$R^iF(C) \xrightarrow{\delta^i} R^{i+1}F(A) \longrightarrow R^{i+1}F(B) \longrightarrow \dots$$

(5) For any injective object I, $R^iF(I) = \{0\}$ for all $i > 0$.

The proof of (1) and (4) requires some substantial effort, although (3) is easy: As F is left exact, the sequence $0 \to F(A) \to F(I^0) \to F(I^1)$ is exact and thus $F(A^0) = \mathrm{Ker}(F(I^0) \to F(I^1)) = R^0F(A)$.

An object J for which $R^iF(J) = \{0\}$ for all $i > 0$ is called F-acyclic. Every injective object is F-acyclic as we may choose the resolution $0 \to I \to I^0 = I \to 0 \to 0 \to \dots$ to compute its cohomology. For any F as above, in fact. Now, another argument in homological algebra gives the following. If $0 \to A \to J^0 \to J^1 \to J^2 \to \cdots$ is a resolution of A by F-acyclic objects then

$$R^i F(A) = \frac{\mathrm{Ker}(F(J^i) \to F(J^{i+1}))}{\mathrm{Im}(F(J^{i-1} \to F(J^i)))}.$$

Thus, it is not necessary to use injective resolutions to calculate the functors $R^i F$. This will be significant below when we compare two ways to calculate the cohomology of sheaves. We end this section be giving two examples of functors that are covariant, left-exact and additive.

Lemma 5.3.1. *Let X be a scheme. The functor of global section*

$$\Gamma : \mathfrak{MOD}(X) \to \mathbf{Ab}.\mathbf{Gps}., \qquad \mathscr{F} \mapsto \Gamma(X, \mathscr{F}),$$

is an additive left-exact covariant functor. We also denote $R^i\Gamma(-)$ by $H^i(X, -)$.

Let $f : X \to Y$ be a morphism of schemes. The functor

$$f_* : \mathfrak{MOD}(X) \to \mathfrak{MOD}(Y)$$

is an additive left-exact covariant functor.

Proof. The only issue is to check that these functors are left exact. Let

$$0 \longrightarrow \mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C} \longrightarrow 0$$

be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. Recall that that means that $\mathscr{A}(U) = \mathrm{Ker}(\mathscr{B}(U) \to \mathscr{C}(U))$ for every open set U and in particular α is injective on each $\mathscr{A}(U)$. Also, as $\mathrm{Ker}(\mathscr{B}(U) \to \mathscr{C}(U))$ is already a sheaf, equal to the sheafification of $U \mapsto \mathscr{A}(U)$, which is already a sheaf, we have $\mathscr{A}(U) = \mathrm{Ker}(\mathscr{B}(U) \to \mathscr{C}(U))$ for all U. In particular for $U = X$. Thus, the sequence

$$0 \to \mathscr{A}(X) \to \mathscr{B}(X) \to \mathscr{C}(X)$$

is exact, as was to be shown.

Now for f_*. We recall that the pair (f^*, f_*) is an adjoint pair ([H] II.5, page 110) of functors between abelian categories. Therefore, f_* is left-exact (and f^* is right-exact). $\square$

5.4. **Flasque sheaves.** Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space X. We say that $\mathscr{F}$ is <u>flasque</u> if for every inclusion of open sets $V \subseteq U$ the restriction map

$$\mathscr{F}(U) \to \mathscr{F}(V)$$

is surjective.

Theorem 5.4.1. *Let X be a ringed space. Let $\mathscr{F}$ be a flasque sheaf of $\mathcal{O}_X$-modules then $\mathscr{F}$ is Γ-acyclic. That is,*

$$H^i(X, \mathscr{F}) = 0, \quad \forall i > 0.$$

Proof. Let $\mathscr{I}$ be an injective object of $\mathfrak{MOD}(X)$ into which $\mathscr{F}$ embeds and let $\mathscr{G} = \mathscr{I}/\mathscr{F}$.

Lemma 5.4.2. *The sheaves $\mathscr{I}$ and $\mathscr{G}$ are flasque.*

Lemma. Denote by j the inclusion map $j : V \to U$. We have the operation $j_!$ of extension by zero from sheaves on V to sheaves on U. Namely, if $\mathscr{J}$ is a sheaf on V we define $j_!\mathscr{J}$ to be the sheaf on U associated to the presheaf with the values

$$W \mapsto \begin{cases} 0 & W \not\subseteq V \\ \mathscr{J}(W) & W \subseteq V. \end{cases}$$

Exercise 5.4.3. Prove that

$$(j_!\mathscr{J})_P = \begin{cases} 0 & P \notin V \\ \mathscr{J}_P & P \in V. \end{cases}$$

and that $j_!\mathscr{J}$ restricted to V is $\mathscr{J}$.

Now, apply this construction twice. First, extend by zero $\mathcal{O}_X\mid_U$ to X and secondly extend by zero $\mathcal{O}_X\mid_V$ to X. Call these sheaves $i_!\mathcal{O}_U$ and $i_!\mathcal{O}_V$, respectively. Note that there is a natural map of sheaves

$$0 \to i_!\mathcal{O}_V \to i_!\mathcal{O}_U,$$

and this sequence is exact, as we quickly using stalks. Thus, as $\mathscr{I}$ is injective, we have an exact sequence

$$\mathrm{Hom}(i_!\mathcal{O}_U, \mathscr{I}) \to \mathrm{Hom}(i_!\mathcal{O}_V, \mathscr{I}) \to 0. \tag{2}$$

We claim that

$$\mathrm{Hom}(i_!\mathcal{O}_U, \mathscr{I}) = \mathscr{I}(U).$$

Indeed, given $\varphi \in \mathrm{Hom}(i_!\mathcal{O}_U, \mathscr{I})$ we have $\varphi_U(1) \in \mathscr{I}(U)$, where here 1 is the identity element of $i_!\mathcal{O}_U(U) = \mathcal{O}_X(U)$. Note that $\varphi_U(1)$ determines not only the map φ_U by $\mathcal{O}_X(U)$-linearity, but by functoriality all the maps φ_W for $W \subseteq U$ and so (examine behaviour on stalks) the map φ. This allows us also to build the converse map, associating to an element t of $\mathscr{I}(U)$ the unique map φ such that $\varphi_U(1) = t$. Thus, (2) says that

$$\mathscr{I}(U) \to \mathscr{I}(V) \to 0$$

is exact, and so that $\mathscr{I}$ is flasque.

To show $\mathscr{G}$ is flasque, we consider the exact sequence

$$0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{G} \to 0.$$

We claim that for every open set U, the following sequence is exact:

$$0 \to \mathscr{F}(U) \to \mathscr{I}(U) \to \mathscr{G}(U) \to 0. \tag{3}$$

We only need to show that $\mathscr{I}(U) \to \mathscr{G}(U) \to 0$ is exact, namely, that $\mathscr{I}(U) \twoheadrightarrow \mathscr{G}(U)$.

Let $s \in \mathscr{G}(U)$ and let $\Sigma = \{(V, \sigma) : V \subset U, \sigma \in \mathscr{I}(V), \sigma \mapsto s|_V\}$. Note that by basic properties of maps of sheaves, Σ is non-empty. In fact, for every $P \in U$ there is an element of the form (V, σ) in Σ such that $P \in V$. Σ is naturally ordered by inclusion and satisfies Zorn's lemma and thus has a maximal element (V, σ). Suppose that $V \neq U$ and choose some $P \in U - V$ and an element (V_1, σ_1) of Σ such that $P \in V_1$. We have that $(\sigma - \sigma_1)|_{V \cap V_1} \mapsto 0$. As $\mathscr{F}$ is flasque, we can extend the element $(\sigma - \sigma_1)|_{V \cap V_1}$ to V_1 and use it to modify σ_1, thus achieving that $\sigma = \sigma_1$ on $V \cap V_1$. We can now glue those sections to a section on $V \cup V_1$ that maps to $s|_{V \cup V_1}$ and that's a contradiction to the maximality of (V, σ).

We now conclude that $\mathscr{G}$ is flasque: for $V \subseteq U$ this follows from the following diagram:

$$\begin{tikzcd} \mathscr{I}(U) \arrow[r, two heads] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \mathscr{I}(V) \arrow[r, two heads, "(3)"] & \mathscr{G}(V) \end{tikzcd}$$

□

At this point, we have the exact sequence of flasque sheaves

$$0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{G} \to 0,$$

in which $\mathscr{I}$ is injective, hence Γ-acyclic. Consider the long exact sequence in cohomolgy

$$0 \to \Gamma(\mathscr{F}) \to \Gamma(\mathscr{I}) \to \Gamma(\mathscr{G}) \to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{I}) = 0.$$

Using the result above (see (3)), $\Gamma(\mathscr{I}) \to \Gamma(\mathscr{G})$ is surjective and so it follows that $H^1(X, \mathscr{F}) = 0$. For <u>all</u> flasque sheaves $\mathscr{F}$. Now we have,

$$0 = H^1(X, \mathscr{G}) \to H^2(X, \mathscr{F}) \to H^2(X, \mathscr{I}) = 0.$$

The first term is zero because $\mathscr{G}$ is flasque. The last term is zero because $\mathscr{I}$ is Γ-acyclic. Thus, also $H^2(X, \mathscr{F}) = 0$ for <u>all</u> flasque sheaves $\mathscr{F}$. Arguing thus by induction we get that $H^i(X, \mathscr{F}) = 0$ for all $i > 0$. □

Remark 5.4.4. Theorem 5.4.1 applies for sheaves of abelian groups on a topological space X. Indeed, we can always endow X with the constant sheaf $\underline{\mathbb{Z}}$, making it a ringed space and making all abelian sheaves into sheaves of $\mathcal{O}_X$-modules.

Theorem 5.4.5 (Grothendieck)**.** *Let $X = \operatorname{Spec} A$ be an affine scheme, where A is a noetherian ring. Then for all $\mathscr{F}$ a quasi-coherent module,*

$$H^i(X, \mathscr{F}) = 0, \quad \forall i > 0.$$

Proof. As $\mathscr{F}$ is quasi-coherent, it is the sheaf associated to the A-module $M = \Gamma(\mathscr{F})$. Choose a resolution of M by injective A-modules

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

Using the equivalence of categories between A-modules and quasi-coherent sheaves, one concludes that the sequence of sheaves

$$0 \to \mathscr{F} = \tilde{M} \to \tilde{I}^0 \to \tilde{I}^1 \to \tilde{I}^2 \to \cdots$$

is an exact sequence of sheaves. However, the sheaves $\tilde{I}^j$, although they are injective objects of $\mathfrak{QCOH}(X)$, they need not be injective in the category $\mathfrak{MOD}(X)$ (which is really where we want to calculate cohomology as natural $\mathcal{O}_K$-modules arising in applications need not be quasi-coherent.) It is therefore not clear if they are even flasque - our proof of "injective $\Rightarrow$ flasque" used sheaves of the form $i_!\mathcal{O}_V$, for an open set V that are rarely flasque.

As it turns out, the sheaves $\tilde{I}^j$ are indeed flasque, but that is a non-trivial point requiring that the ring A is noetherian. See [H] III 3.4 for details. We can therefore compute the cohomology of $\mathscr{F}$ (in $\mathfrak{MOD}(X)$, or even $\mathfrak{AB}(X)$) using the resolution $\tilde{I}^\bullet$ and we find by applying Γ the complex

$$0 \to I^0 \to I^1 \to I^2 \to \dots,$$

which is exact, except at I^0. Hence, $H^i(X, \mathscr{F}) = 0$ for $i > 0$. □

We would be remiss if we didn't mention (without proof, unfortunately) the following theorem of Serre.

Theorem 5.4.6 (Serre; c.f. [H] III 3.7)**.** *Let X be a noetherian scheme. The following are equivalent:*

(1) X is affine;
(2) $H^i(X, \mathscr{F}) = 0$ for all quasi-coherent sheaves $\mathscr{F}$ and $i > 0$.

In addition, we have the following very important theorem.

Theorem 5.4.7 (Grothendieck's vanishing theorem; [H] III, 2.7)**.** *Let X be a noetherian topological space of dimension n. Then, for an abelian sheaf $\mathscr{F} \in \mathfrak{AB}(X)$,*

$$H^i(X, \mathscr{F}) = 0, \quad i > \dim(X).$$

Remark 5.4.8. Although this is a powerful theorem, the proof is not very deep, but is incredibly clever. It uses various reductions and induction arguments that eventually reduce everything to a computation of cohomology of constant sheaves, or their extension by zero.

The first step is reduction to the case where X is irreducible. If Y is an irreducible component of X and $U = X - Y$, one lets $\mathscr{F}_U, \mathscr{F}_Y$ be the extension by zero of the restriction of $\mathscr{F}$ to U and Y respectively to obtain

$$0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Y \to 0.$$

By taking cohomology and making an argument that identifies the cohomology of $\mathscr{F}_Y$ with $\mathscr{F}|_Y$ and $\mathscr{F}_U$ with $\mathscr{F}_{\bar{U}}$ (the topological closure of U of X), one reduces to $\bar{U}$. As $\bar{U}$ has one less irreducible component, eventually one reduces to the case X is irreducible.

Then, an induction argument on the dimension of X takes place. The case of X irreducible of dimension 0 is easy as there are no open sets but X and $\emptyset$. And so $\mathscr{F}$ is just an abelian group and resolution is a resolution as an abelian group, etc.

When X is irreducible of dimension n there is another reduction taking place. One shows that $\mathscr{F}$ is the direct limit of sheaves of the form $\mathscr{F}_\alpha$, where α is a finite set of sections $\alpha_i \in \mathcal{F}(U_i)$ and one takes the sub sheaf of $\mathscr{F}$ generated by those sections. An argument with long exact sequences in cohomology applied to $0 \to \mathscr{F}_\alpha \to \mathscr{F}_{\alpha \cup \alpha_{n+1}} \to \mathscr{F}_{\{\alpha_{n+1}\}} \to 0$, reduces to the case of a sheaf on X generated by one section $\alpha \in \mathscr{F}(U)$. Change notation and call that sheaf $\mathscr{F}$. Then, one has an exact sequence

$$0 \to \mathscr{R} \to \underline{\mathbb{Z}}_U \to \mathscr{F} \to 0.$$

And so one has to study the case of $\underline{\mathbb{Z}}_U$ and its various subsheaves. This is a rather explicit calculation (that uses various tricks; in particular, in examining the various possibilities for $\mathscr{R}$ one is able to use the induction hypothesis on $\dim(X)$), but, so to say, "mystery dispelled".

5.5. **Čech cohomology.**

The discussion here is quite general and applies to any topological space and a sheaf of abelian groups. The calculation of the cohomology by means of Čech cocyles for a given open cover is a powerful and useful technique. The thorny issue is to determine when the calculation actually gives the cohomology of the sheaf as we have previously defined it. We provide a criterion in Theorem 5.7.1.

5.5.1. *The topological setting.* Let X be a topological space and $\{U_i : i \in I\}$ an open cover of X. We assume that the set I is well-ordered (sometimes called 'linearly ordered'). Given $i_0, i_1, \dots, i_p \in I$ with $i_0 < i_1 < \dots < i_p$, we write

$$U_{i_0 i_1 \cdots i_p} = U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_p}.$$

Let $\mathscr{F} \in \mathfrak{AB}(X)$ be a sheaf of abelian groups on X. Define the Čech cocyles of dimension $p \geq 0$ as

$$C^p(\{U_i\}, \mathscr{F}) = \prod_{i_0 < i_1 < \dots < i_p} \mathscr{F}(U_{i_0 i_1 \cdots i_p}).$$

Example 5.5.1. To illustrate, suppose $X = U_0 \cup U_1$ then

C^0	$\mathscr{F}(U_0) \times \mathscr{F}(U_1)$
C^1	$\mathscr{F}(U_0 \cap U_1)$
$C^p, p \geq 2$	0

We shall denote an element α of $C^p(\{U_i\}, \mathscr{F})$ by $(\alpha_{i_0 i_1 \cdots i_p})$. We now define a boundary map

$$d : C^p \to C^{p+1}$$

by

$$(d\alpha)_{i_0 i_1 \cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \cdots \hat{i}_k \cdots i_{p+1}}. \tag{4}$$

It's a standard fact that $d^2 = 0$ (if you have seen homology in a course in topology, it is exactly the same proof).

We have therefore obtained a complex:

$$0 \longrightarrow C^0(\{U_i\}, \mathscr{F}) \xrightarrow{d} C^1(\{U_i\}, \mathscr{F}) \xrightarrow{d} C^2(\{U_i\}, \mathscr{F}) \xrightarrow{d} \dots$$

Definition 5.5.2. The p-th Čech cohomology group of the cover $\{U_i\}$ is

$$\check{\mathrm{H}}^p(\{U_i\}, \mathscr{F}) := H^p(C^\bullet(\{U_i\}, \mathscr{F})) = \frac{\mathrm{Ker}(d : C^p \to C^{p+1})}{\mathrm{Im}(d : C^{p-1} \to C^p)}.$$

Proposition 5.5.3. $\check{H}^0(\{U_i\}, \mathscr{F}) = \Gamma(\mathscr{F}) = \mathscr{F}(X)$.

Proof. This just expresses that fact that given an open cover, the global sections are precisely the sections over the open sets of the cover that agree on intersections. This is part of the definition of a sheaf. □

Example 5.5.4. Referring back to our example we find that for $\alpha = (\alpha_0, \alpha_1)$, $d : C^0 \to C^1$ is given by $(d\alpha)_{01} = \alpha_1 - \alpha_0$. All $d : C^i \to C^{i+1}$ are zero for $i > 0$ due to the simple fact that in our example $C^i = 0$ for $i > 1$.

As we have already remarked (and will prove later) if X is a projective non-singular curve over an algebraically closed field k then $X - \{x_0\}$ is affine for every point $x_0 \in X(k)$. It follows that such X can always be covered by two open affine subsets. Thus, Čech cohomology for this cover of quasi-coherent sheaves, and in fact for any affine cover (using Theorem 5.7.1), vanishes in dimension greater than 1. This is a special case of Theorem 5.4.7.

Remark 5.5.5. At the level of our survey, the definition of Čech cohomology groups for a cover suffices. One can of course wonder if there is a way to get a definition that is independent on the cover by taking some sort of limit over all open covers. This indeed can be done. See [H] III Exercise 4.4 and also 4.5 (for motivation) and 4.11 (for why it is a reasonable construction).

5.6. **Examples of Čech cohomology.** We calculate a few examples, already in the setting that interests us. That is, for certain schemes.

Example 5.6.1. • $\mathscr{O}_X$ for $X = \mathbb{P}^1_k$. We take k to be an algebraically closed field and we take the standard cover of $\mathbb{P}^1_k$ as $U_0 \cup U_1$, where if the coordinates of $\mathbb{P}_1$ are $(x_0 : x_1)$, U_i is the open set $x_i \neq 0$. Thus,

$$U_0 = \mathbb{P}^1_k - \{0\}, \quad U_1 = \mathbb{P}^1_k - \{\infty\}.$$

Writing the function field of $\mathbb{P}^1$ as $k(x)$, where $x = x_1/x_0$ we have

$$\mathcal{O}(U_0) = k[x], \quad \mathcal{O}(U_1) = k[1/x].$$

We find that

$$\mathcal{O}(U_{01}) = F[x][x^{-1}] := \{f(x)/x^d : f(x) \in k[x]\}.$$

Now, $\check{\mathrm{H}}^0(\mathcal{O})$ (where we omit the open cover from the notation) are the global sections of $\mathcal{O}$. We may view those as morphisms $\mathbb{P}^1_k \to \mathbb{A}^1_k$. As $\mathbb{P}^1_k$ is proper and connected, the image is closed and connected, hence constant. That is

$$\check{\mathrm{H}}^0(\mathbb{P}^1_k, \mathcal{O}_k) = \{0\}.$$

We can also see this more directly. The elements in H^0 are $k[x] \cap k[1/x] = k$.

Let us now calculate $\check{\mathrm{H}}^1(\mathbb{P}^1_k, \mathcal{O})$. We need to calculate the quotient

$$\mathcal{O}(U_0 \cap U_1)/\{\alpha_1 - \alpha_0 : \alpha_i \in \mathcal{O}(U_i)\}.$$

We have $\mathcal{O}(U_0 \cap U_1) = \{f(x)/x^d : f(x) \in k[x]\}$ and we need to find the quotient by the group $\{g(x) - h(1/x) : g, h \in k[x]\}$. The elements h appearing there are precisely the quotients $r(x)/x^d$: $r(x) \in k[x], \deg(r) \leq d$. Thus, if the degree of h is d we get $g(x) - h(1/x) = (x^d g(x) + r(x))/x^d$ and so we find all polynomials in $\mathcal{O}(U_0 \cap U_1)$. It follows that

$$\check{\mathrm{H}}^1(\mathbb{P}^1_k, \mathcal{O}) = \{0\}.$$

• **$\Omega_{X/k}$ for $X = \mathbb{P}^1_k$.** Let us now consider the cohomology of the sheaf of differentials $\Omega := \Omega_{\mathbb{P}^1_k/k}$. We note that $U_0 \cong \operatorname{Spec} k[s], s = \frac{x_1}{x_0}, U_1 \cong \operatorname{Spec} k[t], t = \frac{x_0}{x_1}$ are both the affine space. Thus,

$$\Omega(U_0) = k[s] \cdot ds, \quad \Omega(U_1) = k[t] \cdot dt, \quad \Omega(U_0 \cap U_1) = k[s^{\pm 1}] \cdot ds,$$

where for the last equality we used localization. We have

$$\check{\mathrm{H}}^0(\mathbb{P}^1_k, \Omega) = \{(\alpha_0, \alpha_1) : \alpha_i \in \Omega(U_i), \alpha_0|_{U_0 \cap U_1} = \alpha_1|_{U_0 \cap U_1}\}.$$

Writing $\alpha_1 = f(s)ds, \alpha_2 = g(t)dt$ we get that $\alpha_2 = g(1/s)d(1/s) = -g(1/s)s^{-2}ds$ on $U_0 \cap U_1$, which is never of the form $f(s)ds$ unless $f = g = 0$. Thus,

$$\check{\mathrm{H}}^0(\mathbb{P}^1_k, \Omega) = \{0\}.$$

For $\check{\mathrm{H}}^1(\mathbb{P}^1_k, \Omega)$ we need to mod out the differentials $k[s^{\pm 1}] \cdot ds$ by differentials of the form $(-g(1/s)s^{-2} - f(s))ds$. We conclude an isomorphism,

$$\check{\mathrm{H}}^1(\mathbb{P}^1_k, \Omega) \cong k, \qquad f(s)ds \mapsto \operatorname{res}(f(s)ds).$$

Here $f(s)$ is of the form $\sum_{i=-a}^{b} a_i s^i, a_i \in k$ and the residue of the differential $f(s)ds$ at zero is $\operatorname{res}(f(s)ds) = a_{-1}$.

Exercise 5.6.2. Assume for simplicity that the base field k is algebraically closed of characteristic zero. Calculate the zero-th and first cohomology of the projective non-singular plane curve

$$C : x^3 + y^3 + z^3 = 0,$$

for the sheaves $\mathcal{O}_C, \Omega_{C/k}$, using the affine cover of C induced from the standard affine cover of $\mathbb{P}^2_k$ by three copies of $\mathbb{A}^2_k$ (note that C is in fact covered already by any two of these three open sets, which simplifies the calculations). We provide some hints: (i) The dimension of all these cohomology groups is 1. (ii) Note that choosing an affine model $s^3 + t^3 + 1 = 0$, where $s = x/z, t = y/z$ any differential on C can be written as $f(s,t)ds$, with $f(s,t) \in k(C)^\times$. (iii) Show that the differential $\omega := t^{-2}ds = -s^{-2}dt$ is a holomorphic global differential and calculate its divisor. Namely, for every point $P \in C$, choose a local uniformizer at P, say w_P and express this differential in the local ring as $g \cdot dw_P$ and find the valuation of g. (iv) Using this, show that any other non-zero holomorphic differential is a scalar multiple of ω.

Exercise 5.6.3. Calculate $\check{\mathrm{H}}^1(\mathbb{A}^2_k - \{0\}, \mathcal{O}_X)$ using the cover $x \neq 0$ and $y \neq 0$ (that are both affine). Show that it is not zero. More precisely, show that it is isomorphic $\bigoplus_{i,j<0} k \cdot \frac{1}{x^i y^j}$. Using that this Čech cohomology actually calculates $H^1(\mathbb{A}^2_k - \{0\}, \mathcal{O}_X)$, and comparing with Theorem 5.4.6, conclude again that $\mathbb{A}^2_k - \{0\}$ is not affine.

Example 5.6.4. *Skyscrapers sheaves and their cohomology.*

Let X be a scheme and $Z \subset X$ a closed subscheme. Let $\mathscr{F}$ be a sheaf on Z and let $i : Z \to X$ be the closed immersion. The sheaf i_*Z is a sheaf on X and has the property that $(i_*\mathscr{F})_P$ is $\{0\}$ if $P \notin Z$ and is $\mathscr{F}_P$ if P is in Z. Moreover, if $\mathscr{F}$ is a $\mathcal{O}_Z$-module then $i_*\mathscr{F}$ is a sheaf of $\mathcal{O}_X$-modules. (It is naturally a sheaf of $i_*\mathcal{O}_Z$-modules, but $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathscr{I}$, where $\mathscr{I}$ is the quasi-coherent ideal sheaf on X defining Z.) Cf. [H] Exercises 1.17, 1.19, 1.21.

Now suppose that Z is a point on X. For any abelian group A, viewed as the constant sheaf on Z, we have the sheaf $i_P(A)$, where i_P is really i_* for the closed immersion $i : \{P\} \to X$. In particular, we have the structure sheaf of $\{P\}$, which is just $\boldsymbol{k}(P)$ (the residue field of the local ring of P). The sheaf $i_P(\boldsymbol{k}(P))$ (or more generally $i_P(A)$) is called a <u>skyscraper sheaf</u>. It has the property that $i_P(A)(U) = A$ if $P \in U$,and is otherwise 0. In particular, it has zero stalks at every point different from P and its stalk at P is A. Cf. [H] II, Exercises 1.17, 1.19, 1.21.

Now lets look at all that for a variety X over an algebraically closed field k and P a closed point of X; its residue field is k. For every open subset U such that $P \notin U$ we have $i_P(k) = 0$ and the ideal sheaf defining P, say $\mathscr{I}_P$ (a dangerous notation...) satisfies $\mathscr{I}_P(U) = \mathcal{O}_X(U)$. on the other hand, let U be an open affine subset, $U = \mathrm{Spec}(R)$, such that $P \in U$. Then $\mathscr{I}_P$ corresponds to a maximal ideal $\mathfrak{m}$ of R; its value on U is just $\mathfrak{m}$. The value of $i_P(k)$ on U is just k and the short exact sequence $0 \to \mathscr{I}_Z \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$, valid for any closed immersion $i : Z \to X$, specializes for $Z = \{P\}$ and X as above over the set U to $0 \to \tilde{\mathfrak{m}} \to \tilde{R} \to i_P(k) \to 0$ and by passing to global sections to $0 \to \mathfrak{m} \to R \to k \to 0$.

Let X be a variety over k (for simplicity). We claim that

$$\check{\mathrm{H}}^i(X, i_P(k)) = \begin{cases} k & i = 0, \\ \{0\} & i > 0. \end{cases}$$

To prove that, take an affine open cover $\{U_0, U_1, \dots, U_n\}$ of X such that $P \in U_0$ and $\cup_{i=1}^n U_i = X - \{P\}$; such exists since X is quasi-compact. Note that for $p > 0$ the complex C^p is identically zero as $i_P(k)(U) = 0$ for any open set not containing p. The complex C^0 is simply $k \times \{0\} \times \cdots \times \{0\}$ and $d : C^0 \to C^1$ must be the zero map. We get that $\check{\mathrm{H}}^0(X, i_P(k)) = k$. We could have used also Proposition 5.5.3 to deduce that. We remark that Exercise 5.7.2 also yields this computation immediately.

5.7. **Cohomology and Čech cohomology.**

The following theorem is very important. It allows one to calculate cohomology of quasi-coherent sheaves in examples. We have already seen some.

Theorem 5.7.1. *Let X be a noetherian separated scheme. Let $\{U_i\}$ be an open affine cover of X and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then,*

$$\check{H}^i(\{U_i\}, \mathscr{F}) \cong H^i(X, \mathscr{F}).$$

Proof. We will need several facts. Most were already proven, the others are left as exercises.

- Let X be a separated scheme, U_1, U_2 open affine subsets of X then $U_1 \cap U_2$ is affine. (Exercise; cf. [H] II Exercise 4.3)
- $\check{\mathrm{H}}^0(\{U_i\}, \mathscr{F}) \cong H^0(X, \mathscr{F})$. (Already proven).
- Every quasi-coherent sheaf $\mathscr{F}$ embeds in a quasi-coherent flasque sheaf $\mathscr{I}$. ([H] III 3.6)
- Cohomology in the category of quasi-coherent sheaves, or $\mathcal{O}_X$-modules, or abelian sheaves can be calculated using resolutions by flasque sheaves. The cohomology groups of a flasque sheaf $\mathscr{I}$, $H^i(X, \mathscr{I})$, vanish for $i > 0$ (Theorem 5.4.1). It is also a fact that $\check{\mathrm{H}}^i(X, \mathscr{I}) = 0$ for $i > 0$ ([H] III 4.3).

We consider thus a resolution of $\mathscr{F}$,

$$0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{G} \to 0,$$

where $\mathscr{I}$ is flasque and quasi-coherent (and so also $\mathscr{G}$ is quasi-coherent). For every $U = U_{i_0 i_1 \cdots i_p}$ we have the initial part of the long exact sequence in cohomology:

$$0 \to \mathscr{F}(U) \to \mathscr{I}(U) \to \mathscr{G}(U) \to H^1(U, \mathscr{F}|_U) = 0,$$

where the latter is zero because U is affine, being an intersection of affine open subsets, and $\mathscr{F}$ is quasi-coherent. Taking the product, we conclude that the following sequence is exact:

$$0 \to C^\bullet(\{U_i\}, \mathscr{F}) \to C^\bullet(\{U_i\}, \mathscr{I}) \to C^\bullet(\{U_i\}, \mathscr{G}) \to 0.$$

As $\mathscr{I}$ is flasque $\check{\mathrm{H}}^1(X, \mathscr{I}) = 0$ and so, taking the cohomology of the sequence, we get an exact sequence

$$0 \to \Gamma(\mathscr{F}) \to \Gamma(\mathscr{I}) \to \Gamma(\mathscr{G}) \to \check{\mathrm{H}}^1(X, \mathscr{F}) \to 0.$$

As well,

$$\check{\mathrm{H}}^p(X, \mathscr{G}) \cong \check{\mathrm{H}}^{p+1}(X, \mathscr{F}), \qquad \forall p \geq 1.$$

The same holds with usual cohomology:

$$0 \to \Gamma(\mathscr{F}) \to \Gamma(\mathscr{I}) \to \Gamma(\mathscr{G}) \to H^1(X, \mathscr{F}) \to 0,$$

and

$$H^p(X, \mathscr{G}) \cong H^{p+1}(X, \mathscr{F}), \qquad \forall p \geq 1.$$

We conclude that $H^1(X, \mathscr{F}) \cong \check{\mathrm{H}}^1(X, \mathscr{F})$ and we then argue by induction on p that $H^p(X, \mathscr{F}) \cong \check{\mathrm{H}}^p(X, \mathscr{F})$. At the induction step we use that $\mathscr{G}$ is also quasi-coherent and thus $H^{p+1}(X, \mathscr{F}) \cong H^p(X, \mathscr{G}) \cong \check{\mathrm{H}}^p(X, \mathscr{G}) \cong \check{\mathrm{H}}^{p+1}(X, \mathscr{F})$. $\square$

Exercise 5.7.2. Let $f : X \to Y$ be an affine morphism of noetherian separated schemes over k. Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Prove that

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F}).$$

(On a separated scheme the intersection of affine subsets is affine. For the notion of affine morphisms see [H] II, Exercise 5.17. You may freely use it.) Here are two cases where this exercise applies: (i) The closed immersion of a point of X into X; (ii) Any non-constant morphism between projective, possibly singular, curves (any finite morphism of schemes is affine).

5.8. **Some key theorems in cohomology.** The following theorems are important and deep. Unfortunately, we cannot cover their proof in this course. See [H] for the proofs.

Definition 5.8.1. Let X be a non-singular n-dimensional variety over an algebraically closed field k. The canonical sheaf (or dualizing sheaf) of X is

$$\omega_X = \bigwedge^n \Omega_{X/k}.$$

Note that ω_X is a line bundle.

Theorem 5.8.2 (Serre's duality)**.** *Let X be a non-singular n-dimensional variety over an algebraically closed field k. For every locally free sheaf $\mathscr{F}$ on X there is a canonical isomorphism*

$$H^i(X, \mathscr{F}) \cong H^{n-i}(X, \mathscr{F}^* \otimes \omega_X).$$

Example 5.8.3. Suppose that X is a curve. Then, for every invertible sheaf $\mathscr{F}$ we have

$$\Gamma(X, \mathscr{F}) \cong H^1(X, \mathscr{F}^* \otimes \Omega_{X/k}).$$

In particular, taking $\mathscr{F} = \mathcal{O}_X$ we find that

$$k \cong \Gamma(X, \mathcal{O}_X) \cong H^1(X, \Omega_{X/k})$$

and that

$$\Gamma(X, \Omega_X) \cong H^1(X, \mathcal{O}_X).$$

Compare this with the calculation of the dimensions of these cohomology groups carried out in Example 5.6.1

Definition 5.8.4. The genus of a non-singular projective curve X is, alternatively, $\dim_k(\Gamma(X, \Omega_X))$ (the dimension of the global holomorphic differentials on X) or $\dim_x(H^1(X, \mathcal{O}_X))$.

Theorem 5.8.5 (Serre)**.** *Let X be a projective scheme over k. Let $\mathscr{F}$ be a coherent sheaf over X. Then, for every $i \geq 0$, $H^i(X, \mathscr{F})$ is a finite dimensional k vector space.*

6. Curves

6.1. **Divisors, Principal divisors, Pic.** Ernesto Mistretta had covered most of what we need here, so we will be rather brief, essentially providing a resumé of his lectures.

Let X be a non-singular quasi-projective variety over k, an algebraically closed field. By an irreducible effective divisor D on X we mean a closed irreducible sub variety of X of codimension 1. We will also refer to this as a primitive divisor. Let ξ_D be the generic point of D then $\mathcal{O}_{X,D} = \mathcal{O}_{X,\xi_D}$ is a regular local ring of dimension one, hence a dvr. Let v_D be the corresponding valuation. If $U = \mathrm{Spec}(A) \subset X$ and $U \cap D \neq \emptyset$ then $U \cap D$ corresponds to a prime ideal $\mathfrak{p}$ (which is the generic point of D) and $\mathcal{O}_{X,D} = A_{\mathfrak{p}}$.

Fact: D is locally principal. That means that $X = \cup_i U_i$, each U_i open affine, $U_i = \mathrm{Spec}(A_i)$ and $D \cap U_i = \{f_i = 0\}$ for some $f_i \in A_i$. This implies, but is stronger than, $f_i A_{\mathfrak{p}_i} = \mathfrak{p}_i A_{\mathfrak{p}_i}$, where $D \cap U_i$ is defined on A_i by the prime ideal $\mathfrak{p}_i$.

A divisor on X is a formal sum $\sum_D a_D[D]$, where the summation is over primitive divisors and $a_D \in \mathbb{Z}$ and are zero except for finitely many D. To a function $f \in k(X)$, $f \neq 0$, we can associate a divisor

$$\mathrm{div}(f) = (f) = \sum_D v_D(f)[D].$$

Such divisors are called principal divisors. They form a group denoted $\mathrm{Prin}(X)$.

Fact: This is a divisor. Namely, $v_D(f) = 0$ except for finitely many D.

We say that $D_1 \geq D_2$ if $D_1 - D_2 = \sum_D c_D[D]$ and all $c_D \geq 0$.

Fact: f is a regular function on X iff $(f) \geq 0$. Furthermore, that much is also true on any non-empty open set U of X.

For X a projective variety, we get an exact sequence of groups

$$0 \to k(X)^\times/k^\times \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0,$$

where we define $\mathrm{Pic}(X) = \mathrm{Div}(X)/\mathrm{Prin}(X)$.

Example 6.1.1. $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$.

Let H be the hyperplane $x_n = 0$. We show that H is a generator for $\mathrm{Pic}(\mathbb{P}^n)$. Let D be a primitive divisor, then $D = \{f(x_0, \dots, x_n) = 0\}$ for some irreducible homogenous polynomial of degree d (View D in $\mathbb{A}^{n+1}$; as $k[x_0, \dots, x_n]$ is a UFD any prime ideal of height 1 is principal. Thus, in $\mathbb{A}^{n+1}$, D is defined by a polynomial $f(x_0, \dots, x_n)$, which is necessarily homogenous because it has the property that $f(\alpha_0, \dots, \alpha_n) = 0$ if and only if $f(\lambda\alpha_0, \dots, \lambda\alpha_n) = 0$ for all $\lambda \in k$.) Thus, $D = (x_n^{-d} f(x_0, \dots, x_n)) + dH$.

It remains to show that for every $d \geq 1$, dH is not principal. But if $dH = (g)$ for some non-zero function g, then g is a regular on $\mathbb{P}^n$, hence a constant, and vanishes along H. This is a contradiction.

Example 6.1.2. Let X be a non-singular projective curve. Then there is a surjective group homomorphism

$$\mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z}, \qquad \deg(\sum a_D[d]) = \sum a_D.$$

It is well-defined because on a projective curve, $\deg((f)) = 0$ for every function f. One lets $\mathrm{Pic}^0(X)$ be the kernel of the degree map; it is a subgroup of $\mathrm{Pic}(X)$. It is a hard theorem that $\mathrm{Pic}^0(X)$ is the k points of a projective connected algebraic group (a so-called abelian variety) of dimension equal to $g(X)$, the genus of X; it is called the Jacobian variety of X.

6.2. **Locally free sheaves.** We had an impromptu discussion about locally free sheaves, how to describe them using Cech cocyles $\check{\mathrm{H}}^1(X, \mathrm{GL}_n(\mathcal{O}_X))$ and natural operations on sheaves, such as tensor products, direct sum, exterior products, dual. In particular, isomorphism class of <u>invertible sheaves</u>, namely, the locally free sheaves of rank 1 form an abelian group isomorphic to $\check{\mathrm{H}}^1(X, \mathcal{O}_X^\times)$. Under the correspondence, the group operations on invertible sheaves are

- Multiplication: $\mathscr{F}_1 \otimes_{\mathcal{O}_X} \mathscr{F}_2$.
- Identity: $\mathcal{O}_X$.
- Inverse: $\mathscr{F}_1^\vee$.

An interesting result is that the group of invertible sheaves is isomorphic to $\mathrm{Pic}(X)$, namely to divisors up to principal divisors.

6.2.1. *From divisors to invertible sheaves.* We associate to a divisor D a locally free sheaf $\mathcal{O}_X(D)$. For an open set U let

$$\mathcal{O}_X(D)(U) = \{f \in k(X) : (f) \cap U \geq -D\}.$$

Namely, on the open set U, the divisor of f is greater or equal to the divisor $-D$ restricted to U, $-(D \cap U)$. This already gives a sheaf of $\mathcal{O}_X$-modules: One has $(f_1 + f_2) \geq (f_1) + (f_2)$ and $(-f) = (f)$ which gives a structure of an abelian group. If $g \in \mathcal{O}_X(U)$ and $f \in \mathcal{O}_X(D)(U)$ then $(gf) = (g) + (f) \geq (f)$ and so we get an $\mathcal{O}_X$-module structure.

If $D_1 = D + (g)$ there is an isomorphism

$$\mathcal{O}_X(D_1) \xrightarrow{\times g} \mathcal{O}_X(D), \qquad f \mapsto gf.$$

Indeed, suppose $(f) \geq -D_1$ on an open set U then $(fg) = (f) + (g) \geq -D_1 + (g) = -D$.

This also shows that $\mathcal{O}_X(D)$ is locally principal. Indeed, locally D is the divisor of a function g. The argument above shows that multiplication by g^{-1} gives an isomorphism $\mathcal{O}_X \to \mathcal{O}_X(D)$ on that open set.

6.2.2. *From invertible sheaves to divisors.* Let $\mathscr{F}$ be an invertible sheaf. Choose a non-empty open set U and an isomorphism $\varphi : \mathscr{F}|_U \cong \mathcal{O}_U$. Then the function 1 corresponds to a section s of $\mathscr{F}$ that does not vanish on U. The section s extends to a meromorphic section on X. That is, suppose that U_1 is another open set and $\varphi_1 : \mathscr{F}|_{U_1} \cong \mathcal{O}_{U_1}$ is a trivialization. Then $s|_{U\cap U_1}$ corresponds to an invertible function in $\mathcal{O}(U \cap U_1)$ that extends to a meromorphic function f_{U_1} in $k(X)$. We extend the section s to U_1 by $f_{U_1}s$. In such a way we can associate to s a divisor (s). This divisor is trivial on U and on U_1 is the divisor of f_{U_1}. We note that the functions f_{U_1} depend on the choice of trivialization, but nonetheless, the divisor (s) we get this way by covering X by open sets U_1 is well-defined, because the transition maps between different open sets are given by regular functions that are none-vanishing on the intersections hence do not affect the divisor. Denote by $D = (s)$.

Let $t \in \mathscr{F}(V)$. Assume for simplicity that $\mathscr{F}$ is trivial on V (else pass to cover, etc.) then t/s corresponds to a function on V that is independent of the trivialization chosen. This function h satisfies on V that $(h) = (t/s) = (t) - (s) \geq -D$. That is $h \in \mathcal{O}_X(D)$. The converse also holds and we conclude:

Let s be a meromorphic section of $\mathscr{F}$, $D = (s)$ the divisor of s. Then

$$\mathscr{F} \cong \mathcal{O}_X(D).$$

In this was we have passed from invertible sheaves to divisors.

6.2.3. *Functoriality.* It remains to check that $\mathcal{O}_X(D_1+D_2) \cong \mathcal{O}_X(D_1)\otimes_{\mathcal{O}_X}\mathcal{O}_X(D_2)$ and $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^{-1}$. This is not hard to verify from the definitions.

6.3. **Linear systems.** We content ourselves here mainly with definitions. The concept of a linear system is very useful and geometrically enlightening, but we will see that later.

Let D_0 be a divisor on X, where as usual X is a projective non-singular variety. There is a bijective map

$$\mathbb{P}(\Gamma(X, \mathcal{O}_X(D_0))) \Leftrightarrow \{D \geq 0 : D \sim D_0\},$$

given by

$$f \mapsto (f) + D_0.$$

(Here we use $\sim$ to denote that two divisors are equal in $\mathrm{Pic}(X)$, namely, they differ by a principal divisor; this is also called "linearly equivalent"). In particular, the system of effective divisors linearly equivalent to D_0 has a structure of a projective space of dimension $h^0(X, \mathcal{O}_X(D_0)) - 1$. This is called a <u>complete linear system</u> and one denotes it by $|D_0|$. A linear system $\mathfrak{d}$ is by definition a linear subspace of $\mathbb{P}(\Gamma(X, \mathcal{O}_X(D_0)))$. A point P of X is called a <u>base-point</u> of the linear system $\mathfrak{d}$ is any $D \in \mathfrak{d}$ contains P.

6.4. **Examples.**

6.4.1. $X = \mathbb{P}^1 \times \mathbb{P}^1$. Some divisors that are easy to observe are the lines $\{\alpha\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{\alpha\}$ and the diagonal Δ that is equal to the image of the diagonal embedding $\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. Let, for example, D be the divisor $\{0\} \times \mathbb{P}^1$. Writing the function field $k(X)$ as $k(x, y)$, we can write any polynomial $f(x, y)$ in $k[x, y]$ as

$$f(x, y) = \sum_{i \geq 0} a_i(y) x^i, \qquad a_i(y) \in k[y],$$

and then

$$v_D(f) = \min\{i : a_i(y) \neq 0\}.$$

In the same coordinates, the diagonal Δ is defined by $x - y$. That is, $x - y$ is a uniformizer of the local ring $\mathcal{O}_{X,\Delta}$. Choose any unit of that ring, for example $x + y$. Then we can expand a polynomial f as

$$f(x, y) = \sum_i a_i((x + y))(x - y)^i,$$

where the a_i are polynomials and

$$v_\Delta(f) = \min\{i : a_i(y) \neq 0\}.$$

Recall the Segre embedding,

$$\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N, \quad N = (m + 1)(n + 1) - 1,$$

given by

$$(x_0 : \cdots : x_m; y_0 : \cdots : y_m) \mapsto (\cdots : x_i y_j : \cdots).$$

If we let z_{ij} be the coordinates on $\mathbb{P}^N$ then the image is defined by the quadratic equations

$$z_{ij} z_{k\ell} = z_{i\ell} z_{kj}, \qquad \forall i, j, k, \ell,$$

and the Segre embedding is a closed immersion. In particular, we have

$$\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathcal{Q} \hookrightarrow \mathbb{P}^3,$$

where $\mathcal{Q}$ is the quadratic surface

$$\mathcal{Q} : z_{00} z_{11} - z_{01} z_{10} = 0.$$

Fix $(\alpha : \beta) \in \mathbb{P}^1$. The family of lines $\{(\alpha : \beta)\} \times \mathbb{P}^1$ we obtain in $\mathcal{Q}$ are

$$\{\beta z_{00} - \alpha z_{10} = 0\} \cap \mathcal{Q},$$

while the lines $\mathbb{P}^1 \times \{(\alpha : \beta)\}$ are given by

$$\{\beta z_{00} - \alpha z_{01} = 0\} \cap \mathcal{Q}.$$

Taking a homogenous degree polynomial $f(z_{ij})$ we get a divisor on $\mathcal{Q}$ by

$$\mathrm{div}(f) \cap \mathcal{Q}.$$

In the model of $\mathbb{P}^1 \times \mathbb{P}^1$ this correspond to

$$\mathrm{div}(f(x_0 : x_1; y_0 : y_1)),$$

where $f(x_0 : x_1; y_0 : y_1)$ is bi-homogenous of degree d. This divisor can be reducible or irreducible. For example, taking $f = z_{00}z_{11}+z_{01}z_{10}$ that corresponds to $2x_0x_1y_0y_1 = 0$ gives us 4 lines on $\mathbb{P}^1\times\mathbb{P}^1$; taking $f = z_{00}^2+z_{11}^2+tz_{00}z_{01}$ that corresponds to $x_0^2y_0^2+x_1^2y_1^2+tx_0^2y_0y_1$ gives an irreducible divisor, unless $t = 0$. (One way to check that is to first verify irreducibility under $x_1 = y_1 = 1$ and then check that only finitely many points are "hidden" in the complement in $\mathbb{P}^1 \times \mathbb{P}^1$ of this open chart.) For $t = 0$ we get a reducible divisor $(x_0y_0 + ix_1y_1)(x_0y_0 - ix_1y_1) = 0$, whose 2 components intersect at the two points $(1 : 0; 0 : 1), (0 : 1; 1 : 0)$.

6.4.2. $X = \mathbb{P}^2$. Every irreducible divisor D is the zero locus of an irreducible homogenous polynomial $f(X, Y, Z)$. If D_i is defined by $f_i = 0$ and $D_1 \neq D_2$ then the number of intersection points of D_1 and D_2, counted with multiplicities, is $\deg(f_1) \cdot \deg(f_2)$. This is Bezout's theorem.

Let $f, g \in k(\mathbb{P}^2) = k(x, y)$ be the functions $f(x, y) = x/y$ and $g(x, y) = x$. The relation to the homogenous coordinates X, Y, Z is that $x = X/Z, y = Y/Z$. Thus,

$$\mathrm{div}(f) = D_1 - D_2, \qquad \mathrm{div}(g) = D_1 - D_3,$$

where

$$D_1 = \{X = 0\}, \quad D_2 = \{Y = 0\}, \quad D_3 = \{Z = 0\}.$$

6.4.3. $Y = \mathrm{Bl}_P(\mathbb{P}^2)$. Consider now Y, the non-singular projective surface obtained as the blow-up of $\mathbb{P}^2$ at the point $P = (0 : 0 : 1)$. It is birational to $\mathbb{P}^2$ and so $k(Y) = k(x, y)$ as well. What is the divisor of f and g on Y?

Denote $\pi : Y \to \mathbb{P}^2$ the projection and let $E = \pi^{-1}(P)$ be the exceptional divisor. It is irreducible and isomorphic to $\mathbb{P}^1$. We have

$$\pi : Y - E \xrightarrow{\cong} \mathbb{P}^2 - \{P\}.$$

Using the last isomorphism, we see that the real calculation to be done is on $Y^0 = \mathrm{Bl}_0(\mathbb{A}^2) = Y - \tilde{D}_3$ and the functions f, g. From the definitions we get

$$Y^0 = \{(x, y; u : v) : xv = yu\} \subseteq \mathbb{A}^2 \times \mathbb{P}^1.$$

Moreover,

$$E = \{(0, 0; u : v)\}.$$

We have

$$Y^0 = U \cup V,$$

where

$$U = \{(x, y; u : 1) : x = yu\}, \quad E \cap U = \{y = 0\},$$

and

$$V = \{(x, y; 1 : v) : xv = y\}, \quad E \cap V = \{x = 0\}.$$

Remark that $U \cong \mathbb{A}^2_{y,u}$ by $(x, y; u : 1) \mapsto (y, u)$, and under this isomorphism E corresponds to $\{y = 0\}$, f corresponds to the function $(yu)/y = u$ and g to the function yu. Thus, $\mathrm{div}(f) \cap U$ corresponds to $\{u = 0\} = \tilde{D}_1 \cap U$ and $\mathrm{div}(g) \cap U$ corresponds to $E + \{u = 0\} = E + \tilde{D}_1 \cap U$.

Similarly, $V \cong \mathbb{A}^2_{x,v}$ by $(x, y; 1 : v) \mapsto (x, v)$ with E corresponding to $\{x = 0\}$. The function f corresponds to the function $1/v$ and so $\mathrm{div}(f) \cap V = -\{v = 0\} = -\tilde{D}_2$. The function g corresponds to x and $\mathrm{div}(g) \cap V = \{x = 0\} = E$.

We conclude that on Y,

$$\mathrm{div}(f) = \tilde{D}_1 - \tilde{D}_2, \qquad \mathrm{div}(g) = E + \tilde{D}_1 - \tilde{D}_3.$$

6.4.4. *Blow up of the projective plane at six points.* Consider 6 distinct points $\{P_1, \dots, P_6\}$ of the projective plane $\mathbb{P}^2$, such that:

- No 3 points lie on a line;
- Not all points lie on a single conic.

Here by a conic we mean the vanishing locus of any homogeneous non-zero quadratic polynomial $f(x_0 : x_1 : x_2)$. In particular, f may be reduced and even the square of a linear form.

Lemma 6.4.1. *There is a unique conic passing through $P_1, \dots, P_5$. This conic is necessary irreducible.*

Proof. That the conic is irreducible follows from counting. If not, it defines either a union of 2 lines, or a single line (with multiplicity 2) and in either case on one of the lines we would have at least 3 of the points.

We first show that such a conic exists. The space of all quadratic polynomials is a 5-dimensional complete linear system; such a polynomial is given as

$$a_0x^2 + a_1xy + a_2y^2 + a_3xz + a_4yz + a_5z^2,$$

and its divisor is linearly equivalent to the divisor $2D_0$, for example, where $D_0 = \{x_0 = 0\}$. That linear system is then $\mathbb{P}^5$ where we associate to the polynomial the point $(a_0 : \cdots : a_5)$. The condition that a point P_i lies on the conic then translates into a linear condition $\ell_i = 0$ on the coefficients a_i. A conic passing through $P_1, \dots, P_5$ is given by a point on $\ell_1 = \cdots = \ell_5 = 0$ that has dimension at least 0 (and in particular is non-empty!). Indeed, in affine coordinates we intersect 5 hyperplanes $\ell_i = 0$ getting a linear subspace of dimension at least 1 that corresponds to that mentioned point in $\mathbb{P}^5$. That shows that such a conic exists. We next show uniqueness.

Let $\mathscr{C}(P_1, \dots, P_i)$ be the family of conics passing through $P_1, \dots, P_i$. It can be viewed of as the linear system in $\mathbb{P}^5$ given by the vanishing of $\ell_1, \dots, \ell_i$ and we shall refer to its dimension in that sense. Thus, for $i = 0$, it has dimension 5 and for $i = 1$ is has dimension 4 (as it is the subspace of $\mathbb{P}^5$ defined by the vanishing of ℓ_1). We have the following table, that we explain just following it:

	$\mathscr{C}(P_1)$	$\mathscr{C}(P_1, P_2)$	$\mathscr{C}(P_1, P_2, P_3)$	$\mathscr{C}(P_1, \dots, P_4)$	$\mathscr{C}(P_1, \dots, P_5)$
dimension	≤ 4 dim'l	≤ 3 dim'l	≤ 2 dim'l	≤ 1 dim'l	≤ 0 dim'l
member	$L_{13}^2 = 0$	$L_{12}^2 = 0$	$L_{12}L_{13}$	$L_{12}L_{34}$	
not a base point	P_2	P_3	P_4	P_5	

In the table we have used the notation L_{ij} to denote the linear form in 3 variables describing the line in $\mathbb{P}^2$ passing through P_i and P_j. In the first column we note that the linear L_{13}^2 is in the linear system $\mathscr{C}(P_1)$ and that P_2 doesn't line on this line. Thus, the linear system $\mathscr{C}(P_1, P_2)$ is of strictly smaller dimension than that of $\mathscr{C}(P_1)$, giving us the estimate that its dimension is ≤ 3. Similarly, we note that L_{12}^2 is in the linear system $\mathscr{C}(P_1, P_2)$, but that P_3 doesn't lie on it, and therefore the linear system $\mathscr{C}(P_1, P_2, P_3)$ is of smaller dimension; that is, $\mathscr{C}(P_1, P_2, P_3)$ is of dimension at most 2. And so on. □

Let us then take six points $\{P_1, \dots, P_6\}$ satisfying that no 3 are on a line and not all lie on a single conic and blow up $\mathbb{P}^2$ at those points, arriving at the surface

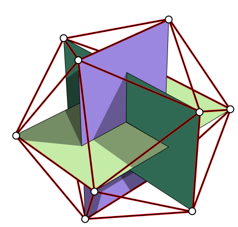
$$X = \mathrm{Bl}_{\{P_1, \dots, P_6\}}(\mathbb{P}^2).$$

We note that X has the following properties:

(1) X is an irreducible non-singular projective surface.
(2) X is birational to $\mathbb{P}^2$.
(3) X has 6 lines given by the exceptional divisors $E_1, \dots, E_6$.
(4) X has $15 = \binom{6}{2}$ lines $\tilde{L}_{ij}$.
(5) X has 6 lines $\tilde{C}_i$, where C_i is the unique conic in $\mathbb{P}^2$ passing all the points P_j except the point P_i.

Here "line" means a non-singular curve of genus 0 and for any $Y \subset \mathbb{P}^2$ we let $\tilde{Y}$ the Zariski closure of $Y - \{P_1, \dots, P_6\}$ in X. A famous theorem (that I hope we will have time to discuss this term) says that every such X is isomorphic to a non-singular cubic surface in $\mathbb{P}^3$ and, conversely, any non-singular cubic surface in $\mathbb{P}^3$ arises this way.

Let us now specialize even further. Consider the regular icosahedron in $\mathbb{R}^3$. The line connecting a vertex of it to the centre passes through precisely one addition vertex. Thus, this give us six lines. We think about these lines as points in $\mathbb{P}^2(\mathbb{R}) \subseteq \mathbb{P}^2(\mathbb{C})$ and we blow up $\mathbb{P}^2(\mathbb{C})$ at these points, getting a surface X called the Clebsch surface. It has several interesting properties:



- As all the blow-ups is it non-singular projective surface birational to $\mathbb{P}^2$.
- It has the model: $\{\sum_{i=0}^4 x_i = 0, \sum_{i=0}^4 x_i^3 = 0\}$ in $\mathbb{P}^4$. This model show that the symmetric group S_5 acts faithfully on X.
- It has the model $\sum_{i=0}^3 x_i^3 - (\sum_{i=0}^3 x_i)^3 = 0$ in $\mathbb{P}^3$. This model, obtained from the one in $\mathbb{P}^4$ by eliminating x_4, exhibits X as a non-singular cubic surface in $\mathbb{P}^3$.

- The action of the icosahedral group A_5 on the icosahedron, which is induced from an orientation preserving linear transformations in $\mathrm{GL}_3(\mathbb{R})$, induced the action of $A_5 \subset S_5$, defined above. If we allow also non-orientation preserving automorphisms of the icosahedron, we get a group isomorphic to $A_5 \times \mathbb{Z}/2\mathbb{Z}$. Indeed, once we know that the orientation preserving automorphisms are A_5, we can choose multiplication by -1 as a non-orientation preserving automorphism[7] and conclude that the full group of automorphisms of the icosahedron is $A_5 \times \{\pm 1\}$. Note however that the projective transformations we get are just the images of A_5 as $\mathrm{diag}(-1,-1,-1)$ is trivial in $\mathrm{PGL}_3(\mathbb{C})$.

  It is known that the full automorphism group of the Clebsch surface is S_5, where the action of A_5 is explained by the icosahedron, but the additional symmetries are not.

6.5. **Euler characteristic.** Let X be a projective variety. Let $\mathscr{F}$ be a coherent $\mathcal{O}_X$ module. Then, by a result of Serre, $H^i(X, \mathscr{F})$ are finite dimensional; we denote the dimension $h^i(X, \mathscr{F})$. We define the Euler characteristic of the sheaf $\mathscr{F}$,

$$\chi(\mathscr{F}) = \sum_{i=0}^{\dim(X)} (-1)^i h^i(X, \mathscr{F}).$$

Lemma 6.5.1. *χ is additive on short exact sequences. That is, if*

$$0 \to \mathscr{F}_1 \to \mathscr{F} \to \mathscr{F}_2 \to 0,$$

is an exact sequence of sheaves, then

$$\chi(\mathscr{F}) = \chi(\mathscr{F}_1) + \chi(\mathscr{F}_2).$$

Proof. Let $d = \dim(X)$. We have a long exact sequence of vector spaces:

$$0 \to H^0(X, \mathscr{F}_1) \to H^0(X, \mathscr{F}) \to \dots \to H^d(X, \mathscr{F}_2) \to 0.$$

As with any long exact sequence of vector spaces, the dimensions counted with alternating signs give zero. Thus,

$$h^0(\mathscr{F}_1) - h^0(\mathscr{F}) + h^0(\mathscr{F}_2) - h^1(\mathscr{F}_1) + \cdots + (-1)^d h^d(\mathscr{F}_2) = 0.$$

But, this just the identity $0 = \chi(\mathscr{F}_1) - \chi(\mathscr{F}) + \chi(\mathscr{F}_2)$. □

Example 6.5.2. Let X be a non-singular projective curve and let $\mathscr{F} = \mathcal{O}_X(D)$, where D is a divisor on X. Then,

$$\begin{aligned}\chi(\mathcal{O}_X(D)) &= h^0(X, \mathcal{O}(D)) - h^1(X, \mathcal{O}_X(D)) \\ &= h^0(X, \mathcal{O}(D)) - h^0(X, \mathcal{O}_X(K - D)).\end{aligned}$$

[7]One knows that the vertices are $(0, \pm 1, \pm\phi)$, where ϕ is the golden ratio and their cyclic permutations, a set that is preserved by multiplication by -1.

In the above, K denotes a canonical divisor and we have used Serre to relate the two expressions for $\chi(\mathcal{O}_X(D))$. Recall how K is defined. One chooses a rational section s of $\omega_{X/k}$, which, since X is a curve, is an invertible sheaf isomorphic to $\Omega_{X/k}$, and one lets $K = \mathrm{div}(s)$. If t is another rational section then t/s is a rational function and so K is defined up to a principal divisor. Let's look at some particular instances.

X **is the projective line** $\mathbb{P}^1$**.** Let x_0, x_1 be the coordinates on the projective line. It contains the affine line $\mathbb{A}^1$ with coordinate $z = x_1/x_0$ and we choose the differential dz. On $\mathbb{A}^1$ the differential dz is regular and non-vanishing. Let $u = x_0/x_1$, then $\mathbb{P}^1 = \mathbb{A}^1_z \cup \mathbb{A}^1_u$, as usual. We have $dz = d(1/u) = -u^-2du$, which has a pole of order 2 at the point "at infinity" $P_\infty = (1:0)$. Thus,

$$(dz) = -2[P_\infty].$$

Note that $H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1/k}) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(K)) = \{f \in k(\mathbb{P}^1) : (f) \geq 2[P_\infty]\} = \{0\}$. Thus, In particular, find that $\chi(\mathbb{P}^1, \mathcal{O}) = 1 - 0 = 1$.

For general information we remark that the general theorem is

$$\chi(\mathbb{P}^n, \mathcal{O}) = 1, \qquad \forall n.$$

X **is the elliptic curve** $y^2z = x^3 + z^3$**.** For simplicity assume that the characteristic is not $2, 3$ so that the curve is non-singular. An affine piece of the curve is given by $X^0 : y^2 = x^3 + 1$. Note that $2ydy = 3x^2dx$. Let ω be the differential form

$$\omega = \frac{dx}{2y} = \frac{dy}{3x^2}.$$

The two expressions together show that ω is a regular differential (there are not points on the curve where both x and y are zero). Moreover, it is non-vanishing (check!). There is but one point on X that is not in X^0, namely the point $(0:1:0)$ at this point we have local coordinates u, v related to x, y by $u = x/y, v = 1/y$ and the curve is given by the equation

$$v = u^3 + v^3.$$

This equation gives us the identity $dv(1 - 3v^2) = 3u^2du$. We calculate, that $2\omega = du - \frac{u}{v}dv$ and then that

$$2\omega = \left(1 - \frac{3u^3}{v(1 - 3u^2)}\right) \cdot du.$$

We have the identity $\frac{u^3}{v} = 1 - v^2$ that shows that at the point $(0:1:0)$ $(= (0,0)$ in the coordinates $u, v)$ the function u^3/v is regular and equal to 1. Using this, we find that $\left(1 - \frac{3u^3}{v(1-3u^2)}\right)$ is regular and non-vanishing at $(0:1:0)$. Thus, we find that the empty divisor is a canonical divisor on X. (This is true, in fact, for any curve of genus 1.) As $H^0(X, \omega_X) \cong H^0(X, \mathcal{O}_X)$, we find

$$\chi(X, \mathcal{O}) = 0.$$

In general, for a non-singular projective curve X:

$$\chi(X, \mathcal{O}_X) = 1 - g(X),$$

by definition of $g(X)$. Thus, our examples above show that $g(\mathbb{P}^1) = 0$ and $g(X) = 1$, where X is the elliptic curve $y^2 = x^3 + 1$.

Example 6.5.3. We now look at the case of surfaces. Let X be a projective non-singular surface. Then,

$$\chi(\mathcal{O}_X(D)) = h^0(O_X(D)) + h^0(\mathcal{O}_X(K - D)) - h^1(\mathcal{O}_X(D)).$$

Let look at the example of $\mathbb{P}^2$ and calculate the canonical divisor for it. In fact, it is equally easy to calculate the canonical divisor for $\mathbb{P}^n$.

Proposition 6.5.4. *Let $X = \mathbb{P}^n$. Let H be a hyperplane of X then $-(n+1)H$ is a canonical divisor.*

Proof. Over the affine open $U_n := \operatorname{Spec} R_n = \operatorname{Spec} k[\frac{x_0}{x_n}, \dots, \frac{x_{n-1}}{x_n}]$ we have

$$\Omega_{U_n/k} = \sum_{i=1}^{n-1} R_n d(\frac{x_i}{x_n}),$$

and

$$\omega_{U_n/k} = \bigwedge^n \Omega_{U_n/k} = R_n \cdot d(\frac{x_0}{x_n}) \wedge \cdots \wedge d(\frac{x_{n-1}}{x_n}).$$

Let $\omega = d(\frac{x_0}{x_n}) \wedge \cdots \wedge d(\frac{x_{n-1}}{x_n})$, viewed as a rational differential form on $\mathbb{P}^n$. The form ω is regular and non-vanishing on U_n. We need to calculate if it has zeros or poles on the complement $H := \mathbb{P}^n - U_n$, which is a primitive divisor. Let $U_0 := \operatorname{Spec} R_0 = \operatorname{Spec} k[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}]$. Similarly, $\Omega_{\mathbb{P}^n/k}|_{U_0} = \oplus_{i=1}^n R_0 \cdot d(\frac{x_i}{x_0}), \omega_{\mathbb{P}^n/k} = R_0 \cdot d(\frac{x_1}{x_0}) \wedge \cdots \wedge d(\frac{x_n}{x_0})$. Note that the complement of $U_0 \cup U_n$ has codimension 2 in $\mathbb{P}^n$ and so "no divisors can hide there". Namely, the only calculation that we need to do is to see whether ω has zeros or poles along H; H is defined in U_0 by the equation $\frac{x_n}{x_0} = 0$.

Writing $\frac{x_i}{x_n} = \frac{x_i}{x_0}(\frac{x_n}{x_0})^{-1}$, one finds that

$$d(\frac{x_i}{x_n}) = (\frac{x_n}{x_0})^{-1} - \frac{x_i}{x_0}(\frac{x_n}{x_0})^{-2} d\left(\frac{x_n}{x_0}\right).$$

Substituting this below, and using $dt \wedge dt = 0$, we find

$$\begin{aligned}
\omega &= d(\frac{x_0}{x_n}) \wedge \cdots \wedge d(\frac{x_{n-1}}{x_n}) \\
&= \left(\frac{x_n}{x_0}\right)^{-n} \left[\left(\frac{x_n}{x_0}\right)^{-1} d\left(\frac{x_n}{x_0}\right) \wedge \left(d\left(\frac{x_1}{x_0}\right) - \frac{x_1}{x_0}\left(\frac{x_n}{x_0}\right)^{-1} d\left(\frac{x_n}{x_0}\right)\right) \wedge \cdots \wedge \left(d\left(\frac{x_{n-1}}{x_0}\right) - \frac{x_{n-1}}{x_0}\left(\frac{x_n}{x_0}\right)^{-1} d\left(\frac{x_n}{x_0}\right)\right)\right] \\
&= \left(\frac{x_n}{x_0}\right)^{-n-1} \cdot d\left(\frac{x_n}{x_0}\right) \wedge d\left(\frac{x_1}{x_0}\right) \wedge \cdots \wedge d\left(\frac{x_{n-1}}{x_0}\right)
\end{aligned}$$

Thus, ω has a pole of order $n+1$ along H. □

Consequently,

$$H^0(\mathbb{P}^n, \omega_{\mathbb{P}^n/k}) = H^0(\mathbb{P}^n, \mathcal{O}(-(n+1)H)) = \{f \in k(\mathbb{P}^n) : (f) \geq (n+1)H\} = 0\},$$

because such a function f has $(f) \geq 0$, in particular, so is a regular constant on $\mathbb{P}^n$, namely a constant; this function also satisfies $(f) \geq (n+1)H$, which is impossible unless $f = 0$.

Now, one can prove using Cech cohomology that $\check{\mathrm{H}}^1(\mathbb{P}^2, \mathcal{O}) = \{0\}$. This might be quite doable using a cover of $\mathbb{P}^2$ with three affine planes, but I hadn't tried. The proof in the general case, which deals with $\mathbb{P}^n$, is in [H], but I suspect that the case of $n = 2$ can be done more simply. At any rate, granted that we find that

$$\chi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 1.$$

6.5.1. *Definition of arithmetic and geometric genus.* Let X be a non-singular projective variety. We define the arithmetic genus of X as

$$p_a(X) = (-1)^{\dim(X)}(\chi(\mathcal{O}_X) - 1);$$

we define the geometric genus of X as

$$p_g(X) = h^0(X, \omega_{X/k}).$$

Here are two examples:

(1) If X is a curve the we saw that $\chi(\mathcal{O}_X) = 1 - g(X)$, where $g(X) = h^0(X, \omega_{X/k})$, by definition. It follows that $p_a(X) = p_g(X) = g(X)$.
(2) IF $X = \mathbb{P}^2$ then by our calculations $p_a(X) = p_g(X) = 0$.

We remark that in general, even for non-singular projective surfaces X, $p_a(X) \neq p_g(X)$.

6.6. **The Riemann-Roch theorem for curves.**

Theorem 6.6.1. *Let X be a non-singular projective curve over an algebraically closed field k. Let D be a divisor on X and g the genus of X. Then,*

$$\chi(\mathcal{O}(D)) = \deg(D) + 1 - g.$$

Note that $\chi(\mathcal{O}(D)) = h^0(X, \mathcal{O}(D)) - h^1(X, \mathcal{O}(D))$ which is equal by Serre's duality to $h^0(X, \mathcal{O}(D)) - h^0(X, \mathcal{O}(K - D))$, where K is a canonical divisor on X. Thus, another formulation of the Riemann-Roch theorem is

$$h^0(X, \mathcal{O}(D)) - h^0(X, \mathcal{O}(K - D)) = \deg(D) + 1 - g.$$

Proof. Begin with the case where D is the empty divisor. The formula then reads

$$h^0(X, \mathcal{O}) - h^1(X, \mathcal{O}) = 1 - g.$$

However, $H^0(X, \mathcal{O}) = k$ and $h^1(X, \mathcal{O}) = g$, by definition. So the formula checks.

We now prove that given a divisor D and a point $P \in X(k)$, the theorem holds true for D if an only if it holds true for $D - [P]$ and that suffices to prove the theorem. More precisely, we show that $\chi(\mathcal{O}(D - [P])) + 1 = \chi(\mathcal{O}(D))$; as $\deg(D - [P]) + 1 = \deg(D)$ that suffices to show that the theorem holds for D iff it holds for $D - [P]$.

Consider the exact sequence

$$0 \to \mathcal{O}(-[P]) \to \mathcal{O} \to i_P(k) \to 0,$$

where $\mathcal{O}(-[P])$ is the ideal sheaf of functions vanishing on $[P]$ to order at least 1 and it defines the closed immersion $i = i_P : P \to X$. Here $i_P(x)$ is the extension by zero of the sheaf of functions k on P. Cf. Example 5.6.4. Tensor this sequence by the sheaf $\mathcal{O}(D)$.

First, note that $\mathcal{O}(D) \otimes i_P(k) = i_P(k)$. Indeed, for an open set U not containing P both sheaves take value 0. For an open set U that contains P, and small enough so that $\mathcal{O}(D)$ is trivial, both sides give the value k. That is, both sheaves have stalks k at the point P.[8]

As $\mathcal{O}_D$ is locally free, and exactness is measured locally, the sequence stays exact and we get

$$0 \to \mathcal{O}(D - [P]) \to \mathcal{O}(D) \to i_P(k) \to 0.$$

Apply the Euler characteristic to get

$$\chi(\mathcal{O}(D - [P])) + \chi(i_P(k)) = \chi(\mathcal{O}(D)).$$

However, we have calculated before that $\chi(i_P(k)) = 1$ (Example 5.6.4) and so the proof is complete.

□

6.7. **Consequences of Riemann-Roch.** Let us apply the Riemann-Roch theorem to the canonical class K. On the one hand,

$$\chi(\mathcal{O}(K)) = h^0(X, \Omega_{X/k}) - h^0(X, \mathcal{O}) = g - 1.$$

On the other hand, Riemann-Roch gives

$$\chi(\mathcal{O}(K)) = \deg(K) + 1 - g.$$

Therefore, quite unexpectedly, we get:

Corollary 6.7.1. $\deg(K) = 2g - 2$.

Exercise 6.7.2. Consider the meromorphic differential dx on the plane curve $x^d + y^d + z^d = 0$, for $d > 0$. Find its divisor K. Calculate the degree of K and conclude that the genus of the curve is $(d-1)(d-2)/2$.

Consider now the case of divisors D of large degree.

[8] One can also use [H], II Ex. 1.19.

Corollary 6.7.3. *Let D be a divisor of degree $\deg(D) > 2g-2$. Then*

$$\dim_k H^0(X, \mathcal{O}(D)) = \deg(D) + 1 - g.$$

Proof. This follows from the fact that if F is a divisor of negative degree then $H^0(X, \mathcal{O}(F)) = \{0\}$, because $\mathrm{div}(f) \geq -D$ implies $\deg(\mathrm{div}(f)) > 0$, which is impossible. We apply it to the divisor $K - D$ and conclude $H^0(X, \mathcal{O}(K-D)) = \{0\}$. By Riemann-Roch $\dim_k H^0(X, \mathcal{O}(D)) = \deg(D)+1-g$. □

Corollary 6.7.4. *Let X be a non-singular projective curve, $n \geq 1$ an integer and $\{x_1, \dots, x_n\}$ distinct points on X. Then $X - \{x_1, \dots, x_n\}$ is an affine curve.*

Proof. We do the case $n = 1$, leaving the general case as an exercise. Consider the divisor $D = (2g+1)[x_1]$, where g is the genus of X. It follows from Corollary 6.7.3 that $\dim H^0(X, \mathcal{O}(D)) = g+2 \geq 2$. Thus, there is a non-constant function on X with a pole at $[x_1]$ and non-where else. It defines a morphism $f : X \to \mathbb{P}^1$. As f is finite flat, we have seen in Corollary 4.6.3 that $f^{-1}(\mathbb{A}^1) = X - \{x_1\}$ is affine. □

Corollary 6.7.5. *Let X be a non-singular projective curve of genus 0 then $X \cong \mathbb{P}^1$.*

Proof. Choose a point $t \in X$ and apply Corollary 6.7.3 to the divisor $[t]$ of degree $1 > 2g(X) - 2$. We have $\dim H^0(X, \mathcal{O}([t])) = 2$ and so there is a function f on X whose polar divisor is precisely $[t]$. We view f as a morphism $f : X \to \mathbb{P}^1$. It is a surjective morphism of degree 1 between non-singular projective curves and so $k(X) = k(\mathbb{P}^1)$ and it follows that $X \cong \mathbb{P}^1$. □

Remark 6.7.6. Recall from the first term our discussion of moduli spaces of curves. The result above says that the coarse moduli space of curves of genus 0 is a single point. It goes back to Riemann that the coarse moduli space of curves of genus 1 (with a marked point) is 1 dimensional curve isomorphic to $\mathbb{A}^1$ (this may be due to Klein) and that the moduli space of curves of genus g is of dimension $3g - 3$, if $g \geq 2$.

6.7.1. *An observation regarding* $\mathrm{Pic}^0(X)$. Let X be a smooth non-singular curve of genus g. Recall that $\mathrm{Pic}^0(X)$ was defined as the group of divisors of degree 0 on X modulo principal divisors. If X has genus 0 then $\mathrm{Pic}^0(X) = \{0\}$ and that's an easy exercise in polynomials of one variable. If X has positive genus g, it is a major theorem that $\mathrm{Pic}^0(X)$ is an abelian connected algebraic group (those are known as abelian varieties), which is projective of dimension g. It is also called the Jacobian variety of X and denoted $\mathrm{Jac}(X)$. People that know Lie groups will be able to conclude that if the ground field k is $\mathbb{C}$ then $\mathrm{Pic}^0(X)(\mathbb{C}) \cong \mathbb{C}^g/\Lambda$ as an analytic Lie group, where Λ is a lattice in $\mathbb{C}^g$ (a discrete abelian subgroup of rank $2g$). Moreover, there is a morphism $X \hookrightarrow \mathrm{Jac}(X)$ which is a closed immersion. We describe it at the level of sets.

Choose a base point $P_0 \in X$ and consider the map

$$\varphi_P : X \to \mathrm{Pic}^0(X), \qquad P \mapsto [P] - [P_0].$$

This is an injective map: suppose that $[P]-[P_0] = Q-[P_0]$ in $\mathrm{Pic}^0(X)$. Then $[P]-[Q] = 0$ there and that means that there is a function $f \in k(X)^\times$ such that $(f) = [P]-[Q]$. As f has one zero, and it's a simple zero, we conclude that the morphism $f : X \to \mathbb{P}^1$ is an isomorphism and that says that X has genus 0, contrary to our assumption.

Note that the map constructed thus is canonical up to the choice of base point P_0. A choice of another base point $[P_1]$ amount to shifting the image of X in $\mathrm{Pic}^0(X)$. To be precise: $\varphi_P + T_{[P_1]-[P_0]} = \varphi_{P_1}$ where $T_{[P_1]-[P_0]}$ is the translation map $T_{[P_1]-[P_0]}(D) = D + [P_1] - [P_0]$.

If one considers the map

$$\varphi_P : X^{g-1} \to \mathrm{Pic}^0(X), \qquad \varphi_P(P_1, \dots, P_{g-1}) = [P_1] + \cdots + [P_{g-1}] - (g-1)[P],$$

after moding out by the action of S_{g-1} on the source, one finds a birational morphism onto the image. The image is an ample divisor on $\mathrm{Pic}^0(X)$ called the theta divisor. It is well-defined up to translation and so up to algebraic equivalence (see Mistretta's lectures for this notion of algebraic equivalence). Altogether this gives a morphism, called the Torelli morphism,

$$\mathcal{M}_g \to \mathcal{A}_g,$$

from the moduli space of projective non-singular curves of genus g to the moduli space of principally polarized abelian varieties of dimension g. We are not going to explain the notion of "principal polarization" but only remark that the theta divisor defines one. The Torelli morphism is known to be a proper and injective (Torelli's theorem) and is in fact a closed immersion in characteristic 0 (and not in general) by a theorem of Oort and Steenbrink.

6.7.2. *The Weierstrass form of an elliptic curve.* Let us consider now a curve E of genus 1. Pick a point $t \in E$ and consider the divisors $[t], 2[t], 3[t]$. The pair (E, t) is called an elliptic curve. Corollary 6.7.3 gives

$$\dim H^0(E, \mathcal{O}([t])) = 1$$
$$\dim H^0(E, \mathcal{O}(2[t])) = 2$$
$$\dim H^0(E, \mathcal{O}(3[t])) = 3$$

As the scalars belong to $H^0(E, \mathcal{O}([t]))$ we conclude that $H^0(E, \mathcal{O}([t])) = k$. Further, we conclude that there is a function on E, call it x such that the polar part of its divisor satisfies $(x)_\infty = -2[t]$. Similarly, there is a function y with $(y)_\infty = -3[t]$. Now consider the functions

$$\{1, x, y, x^2, xy, y^2, x^3\} \subset H^0(E, \mathcal{O}(6[t])).$$

As there are 7 functions and $\dim H^0(E, \mathcal{O}(6[t])) = 6$ there is a linear relation between them. As the functions $1, x, y, x^2, xy$ are linearly independent (consider the valuation of such a linear combination at the point t), and stay so if we throw in either y^2 or x^3, in that linear relation the coefficients of both x^3 and y^2 are not zero. After rescaling, we conclude an equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

We get a rational morphism $E - - > \mathbb{P}^2$, $u \mapsto (x(u), y(u))$ and the image in contained in the projective curve

$$Y : y^2z + a_1xyz + a_3yz^2 - x^3 - a_2x^2z - a_4xz^2 - a_6z^3 = 0. \tag{5}$$

As E is non-singular and Y is a projective curve, we conclude a dominant morphism $E \to Y$.

It remains to show that this is an isomorphism. We first prove it is a birational morphism. We need to show that $[k(E) : k(x, y)] = 1$. However, the function x defines a morphism $x : E \to \mathbb{P}^1$ of degree 2 (as it takes the value ∞ exactly twice) and the function y defines a morphism $y : E \to \mathbb{P}^1$ of degree 3. That is, $[k(E) : k(x)] = 2, [k(E) : k(y)] = 3$ and so $[k(E) : k(x, y)] = 1$. At this point, it will be enough to show that Y is non-singular.

Suppose that Y had a singular point. We will show then that Y is birational to $\mathbb{P}^1$. It then follows that $\mathbb{P}^1$ is isomorphic to E (as they have the same function field $k(x, y)$), which is a contradiction since $\mathbb{P}^1$ has genus 0 and E has genus 1.

Suppose Y has a singular point. By a linear change of coordinates that doesn't change the form of the equation, although it changes the constants a_i, we may assume that the singular point is $(0, 0)$. It follows that $2y + a_1x + a_3$ and $a_1y - (3x^2 + 2a_2x + a_4)$ vanish at $(0, 0)$, which implies $a_3 = a_4 = 0$ and $a_6 = 0$ as well, because $(0, 0)$ is on the curve. Thus, necessarily Y has the form

$$Y : y^2 + a_1xy = x^3 + a_2x^2.$$

Let $t = y/x$. Then $t^2 + a_1t - a_2 = x$ and $tx = y$. That shows that $k(x, y) = k(t)$, and Y is birational to $\mathbb{P}^1$.

We conclude that every elliptic curve (E, t) of genus 1 is isomorphic to a plane curve in Weierstrass form with the point t corresponding the $(0 : 1 : 0)$.

6.8. **The Hurwitz genus formula.** Our purpose in this section is to prove that following theorem (in fact, a more general version of which):

Theorem 6.8.1 (Hurwitz)**.** *Let $f : X \to Y$ be a surjective separable morphism that is tamely ramified between non-singular projective curves over an algebraically closed field k. Then*

$$2g(X) - 2 = \deg(f)(2g(Y) - 2) + \sum_{P \in X} e_P - 1.$$

Recall that e_P is the ramification index at a point P. We will recall the definition below and we shall show that $e_P = 1$ except for finitely many points and so the sum on the right makes sense. To say f is tamely ramified means that if $\mathrm{char}(k) = p > 0$ then $p \nmid e_P$ for all P. Finally, f is separable if $k(X)$ is a separable field extension of $k(Y)$. Thus, at this point, at least the statement of the

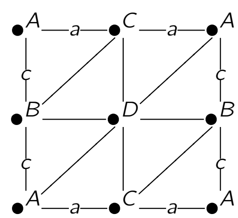
Hurwitz genus formula is understood. Before beginning the proof, we take a detour and explain the situation over the complex numbers.

6.8.1. *The genus formula for complex algebraic curves.* In this section, we will be assuming much we didn't prove. But I think that the intuition gained justifies that.

Over the complex numbers, $X(\mathbb{C})$ and $Y(\mathbb{C})$ are compact complex surfaces, in fact Riemann surfaces. $2g(X)-2$ is minus the topological Euler characteristics. That is, if we triangulate $X(\mathbb{C})$ then the formula

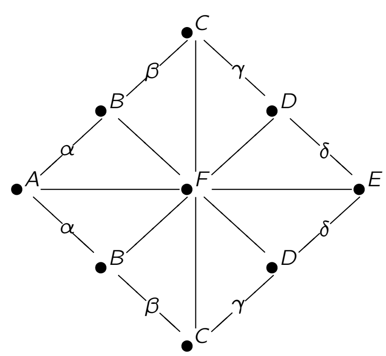
$$2-2g(X) = \sharp \text{ vertices} - \sharp \text{ edges} + \sharp \text{ faces},$$

holds. For example, the following diagram is a triangulation of a torus (a torus has genus 1). One should be glueing opposite sides to get the torus and in particular all vertices marked by the same letter are identified:



The Euler characteristic formula gives $4-12+8=0$.

Similarly, here is a triangulation of a sphere:



The Euler characteristic formula gives $6-12+8=2$.

Now, as $f : X \to Y$ is analytic, if $P \in X$ is a ramification point of order e, one can choose local analytic coordinates at P and $Q = f(P)$ such that the map is given there by $z \mapsto z^e$ (thought of now as a map of the open unit disc to itself). Suppose that Y is triangulated so that the ramification points $R_Y = \{Q_1, \dots, Q_t\}$ of f on Y are among the vertices. We have

$$2-2g(Y) = \sharp V_Y - \sharp E_Y + \sharp F_Y.$$

As f is a covering map $X - f^{-1}(R_Y) \to Y - R_Y$, we get an induced triangulation of X for which

$$\sharp E_X = \deg(f) \cdot \sharp E_Y, \quad \sharp F_X = \deg(f) \cdot \sharp F_Y,$$

but the formula for the vertices must take into account ramification and we find

$$\deg(f)\cdot\sharp\, V_Y - \sharp V_X = \sum_{i=1}^{t}\sum_{f(P)=Q_i}(e_P-1) = \sum_{P\in X}(e_P-1).$$

Therefore,

$$\begin{aligned}\deg(f)(2-2g(Y)) &= \deg(f)(\sharp\, V_Y - \sharp\, E_Y + \sharp\, F_Y)\\ &= \sharp\, V_X - \sharp\, E_X + \sharp\, F_X + \deg(f)\sharp\, V_Y - V_X\\ &= 2-2g(X) + \sum_{P\in X}(e_P-1).\end{aligned}$$

This is Hurwitz's formula.

6.8.2. *Preparations for the algebraic proof.*

Divisors and closed subschemes of curves. Let $D=\sum_{i=1}^t n_i\cdot[P_i]$, $n_i>0$, be an effective divisor on a projective non-singular curve X over an algebraically closed field k. The ideal sheaf $\mathcal{O}_X(-D)$ defines a closed subscheme Z of X, where

$$0\rightarrow\mathcal{O}_X(-D)\rightarrow\mathcal{O}_X\rightarrow\mathcal{O}_Z\rightarrow 0,$$

where as usual $\mathcal{O}_Z$ really means $i_*\mathcal{O}_Z$ with $i:Z\rightarrow X$ the closed immersion. The scheme Z is supported on $\{P_1,\dots,P_t\}$ and

$$\mathcal{O}_Z = \oplus_{i=1}^t i_{P_i}(\mathcal{O}_{P_i}/\mathfrak{m}_{P_i}^{n_i}).$$

Conversely, let Z be a closed non-empty subscheme of X, $Z\neq X$. By definition, for some coherent ideal sheaf I we have

$$0\rightarrow I\rightarrow\mathcal{O}_X\rightarrow\mathcal{O}_Z\rightarrow 0.$$

If $Z_{\rm red}=\{P_1,\dots,P_t\}$ then at $Q\notin\{P_1,\dots,P_t\}$ we have $I_Q=\mathcal{O}_{X,Q}$ (because $\mathcal{O}_{Z,Q}=0$). For $Q=P_i$, I_Q is an ideal of the dvr $\mathcal{O}_{X,Q}$. It cannot be the zero ideal. One checks that in that case $I=0$ in an open neighbourhood of Q which implies that Z contains an open subset of X and so, being closed, $Z=X$. Thus, $I_Q=\mathfrak{m}_Q^{n_i}$ for some $n_i>0$. We recognize that I is nothing but the sheaf $\mathcal{O}_X(-D)$, where $D=\sum_{i=1}^t n_i[P_i]$.

We have set a bijection between closed subschemes $Z\subsetneq X$ and effective divisors D on X.

Pulling back divisors. Let $f:X\rightarrow Y$ be a surjective morphism of projective non-singular curves. Let $Q\in Y$. Recall that if t is a local uniformizer at Q, that is, t is a function in $k(Y)$ such that $t\mathcal{O}_{Y,Q}=\mathfrak{m}_Q$, then for a point $P\in X$ such that $f(P)=Q$, $f^*(t)\cdot\mathcal{O}_{X,P}=\mathfrak{m}_P^{e_P}$ for some positive integer e_P, called the ramification index at P. We proved:

$$\sum_{f(P)=Q} e_P = \deg(f).$$

Define
$$f^*([Q]) = \sum_{f(P)=Q} e_P \cdot [P],$$
and extend by linearity to a homomorphism
$$f^* : \mathrm{Div}(Y) \to \mathrm{Div}(X).$$
This homomorphism has the property
$$\deg(f^* D) = \deg(f)\deg(D).$$
In particular it takes degree zero divisors to degree zero divisor. Evidently, for $g \in k(Y)^\times$ with divisor D we have
$$\mathrm{div}(f^* D) = \mathrm{div}(g \circ f).$$
One concludes induced homomorphisms
$$\mathrm{Pic}(Y) \to \mathrm{Pic}(X), \qquad \mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X).$$

6.8.3. *The proof of Hurwitz's genus formula.* We begin with the following result concerning differentials.

Lemma 6.8.2. *Let $f : X \to Y$ be a separable surjective morphism of non-singular projective curves over k. There is an exact sequence*

$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0. \tag{6}$$

Proof. From general principles (Proposition 1.5.3), we have an exact sequence
$$f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0.$$
It remains to show that the first arrow is injective. But, both $f^*\Omega_{Y/k}$ and $\Omega_{X/k}$ are invertible sheaves on X. In particular, if P is a closed point of X and η its generic point, we have a commutative diagram:
$$\begin{tikzcd}
(f^*\Omega_{Y/k})_P \arrow[r] \arrow[d] & (\Omega_{X/k})_P \arrow[d] \\
(f^*\Omega_{Y/k})_\eta \arrow[r] & (\Omega_{X/k})_\eta
\end{tikzcd}$$
and the vertical arrows are injective. Thus, it is enough to prove that $(f^*\Omega_{Y/k})_\eta \to (\Omega_{X/k})_\eta$ is injective. Under the separability assumption, we have proved (see page 9) that $(f^*\Omega_{Y/k})_\eta = \Omega_{k(Y)/k} \otimes_{k(Y)} k(X) = \Omega_{k(X)/k}$ and $\Omega_{k(X)/k(Y)} = 0$. Thus, at the generic point, we have an exact sequence of $k(X)$-module:
$$k(X) \to k(X) \to 0 \to 0,$$
for which the first arrow must be injective (in fact an isomorphism). $\square$

Under the same assumptions on $f : X \to Y$ we next prove:

Lemma 6.8.3. *The sheaf $\Omega_{X/Y}$ is supported on the ramification points P of X, the points such that $e_P > 1$. In particular, there are finitely many ramification points.*

Proof. Locally on Y, $\Omega_{Y/k}|_U = \mathcal{O}_U \cdot dy$ just because it is locally free. Let $\tilde{U} = f^{-1}(U)$. As we have seen while studying curves, it is also affine. And we may shrink U so that also $\Omega_{X/k}|_{\tilde{U}} = \mathcal{O}_{\tilde{U}} \cdot dx$. On the other hand $f^*(\Omega_{Y/k}|_U) = \mathcal{O}_{\tilde{U}} \cdot f^*dy$ and we may write $f^*dy = h \cdot dx$ for some $h \in \mathcal{O}(\tilde{U})$. Thus, on the open set $U - \mathrm{div}(h)$, we have an isomorphism between $f^*\Omega_Y$ and Ω_X.

Since Y is quasi-compact, it is covered by finitely many such U and so, outside a finite set $f^*\Omega_Y$ and Ω_X are isomorphic. This shows that $\Omega_{X/Y}$ is supported on finitely many points.

More precisely, let $P \in \tilde{U}$ and $Q = f(P)$. In the notation above we may assume that y is a uniformizer at Q and x a uniformizer at P (else replace them by $y - y(Q)$, etc.). Then,

$$f^*dy = d(f^*y) = d(x^{e_P} \cdot u),$$

where u is a unit of $\mathcal{O}_{X,P}$. But,

$$d(x^{e_P} \cdot u) = e_P \cdot x^{e_P - 1} u \cdot dx + x^{e_P} du.$$

Therefore,

$$v_P\left(\frac{f^*dy}{dx}\right) = v_P(h),$$

and, (1) whenever f is tamely ramified at P, which includes all points where $e_P = 1$,

$$v_P\left(\frac{f^*dy}{dx}\right) = e_P - 1;$$

(2) whenever f is wildly ramified,

$$v_P\left(\frac{f^*dy}{dx}\right) \geq e_P.$$

In particular, the sheaf $\Omega_{X/Y}$ is supported exactly at the ramification points. $\square$

Define the ramification divisor R as the effective divisor on X given by

$$\sum_{P \in X} \ell((\Omega_{X/Y})_P) \cdot [P],$$

where $\ell((\Omega_{X/Y})_P)$ is the length as an $\mathcal{O}_{X,P}$-module, which is also its dimension as a k-vector space. Our calculations above show the following:

- R is supported exactly on the ramification points. That is, the point $P \in X$ such that $e_P > 1$.
- At every tamely ramified point P, $\ell((\Omega_{X/Y})_P) = e_P - 1$.
- At every point P, in the notation above, $\ell((\Omega_{X/Y})_P) = v_P(e_P \cdot x^{e_P - 1} u + x^{e_P} \frac{du}{dx})$.
- $\Omega_{X/Y} = \mathcal{O}_X / \mathcal{O}_X(-R)$. We may think about R as a closed subscheme $i : Z \to X$.

The exact sequence in (6) reads then as follows:

$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to i_*\mathcal{O}_Z \to 0.$$

If we tensor by $\Omega_{X/k}^{-1}$ we get

$$0 \to f^*\Omega_{Y/k} \otimes_{\mathcal{O}_X} \Omega_{X/k}^{-1} \to \mathcal{O}_x \to i_*\mathcal{O}_Z \to 0.$$

(The sheaf $i_*\mathcal{O}_Z$ doesn't change; cf. the proof of the Riemann-Roch formula.) Consequently,

$$f^*\Omega_{Y/k} \otimes_{\mathcal{O}_X} \Omega_{X/k}^{-1} \cong \mathcal{O}(-R).$$

But, if $\Omega_{Y/k} \cong \mathcal{O}_Y(K_Y)$ then $f^*\Omega_{Y/k} \cong \mathcal{O}_X(f^*(K_Y))$. And thus $f^*\Omega_{Y/k} \otimes_{\mathcal{O}_X} \Omega_{X/k}^{-1} \cong \mathcal{O}_X(f^*(K_Y) - K_X)$. Taking degrees we find $-\deg(R) = \deg(f^*(K_Y) - K_X) = \deg(f)(2g(Y) - 2) - (2g(X) - 2)$. In conclusion,

$$2g(X) - 2 = \deg(f)(2g(Y) - 2) + \sum_{P \in X} \ell((\Omega_{X/Y})_P).$$

And we have found much information about $\ell((\Omega_{X/Y})_P)$. In particular, if f is tamely ramified,

$$2g(X) - 2 = \deg(f)(2g(Y) - 2) + \sum_{P \in X} (e_P - 1).$$

6.9. **Examples and applications of the Hurwitz genus formula.**

Example 6.9.1. The curve $C : x^d + y^d = z^d$ in $\mathbb{P}_k^2$. Assume that the characteristic of the field k doesn't divide d. Then the curve is non-singular and has genus

$$\frac{(d-1)(d-2)}{2}.$$

Indeed, we have seen in assignments that the morphism $(x : y : z) \to (x : y)$ from C to $\mathbb{P}^1$ is a surjective morphism of degree d (and thus a separable morphism) ramified at d points. At each ramification point P, $e_P = d$. We apply the Hurwitz formula for tamely ramified morphisms to conclude that $2g(C) - 2 = d \cdot (-2) + d(d-1)$, from which the formula follows.

We remark that in fact every non-singular curve of degree d in $\mathbb{P}^2$ has genus $\frac{(d-1)(d-2)}{2}$.

Example 6.9.2. Wild ramification. The situation in the case where the ramification is not tame (one says that the point is then wildly ramified) is more delicate. We provide a simple, yet typical, example.

Let k be an algebraically closed field of characteristic p. Consider the morphism

$$f : \mathbb{P}_k^1 \to \mathbb{P}_k^1, \qquad f(x) = x^p - x.$$

It corresponds to the inclusion of functions fields $k(x) \supseteq k(t)$, where $t = x^p - x$. Thus, $k(x) = k(t)[y]/(y^p - y - t)$ and the polynomial $y^p - y - t$ has derivative relative to y equal to -1. Therefore, this is a separable polynomial and f is a separable morphism. Furthermore, $\deg(f) = p$.

The morphism f is unramified over $\mathbb{A}^1$. Indeed, given $t \in \mathbb{A}^1$ and y solving $y^p - y = t$, for all $i \in \mathbb{F}_p$ we have

$$(y+i)^p - (y+i) = y^p - y + (i^p - i) = t,$$

as well. That is, there are p points lying above t and so the morphism is unramified above any $t \in \mathbb{A}^1(k)$. As there is a unique point left, we conclude that $e_\infty = p$, where ∞ is the point in $\mathbb{P}^1 - \mathbb{A}^1$. Note that the usual Hurwitz formula doesn't hold

$$-2 = 2 \cdot 0 - 2 \neq p \cdot (2 \cdot 0 - 2) + e_\infty - 1 = -p - 1.$$

Indeed, we are in the case of wild ramification: $p | e_\infty$. We must use the ramification divisor and calculate the length of $\Omega_{X/Y}$ at ∞, where $X = Y = \mathbb{P}^1$.

Write $x = X/Y$. The morphism f is given in homogenous coordinates by

$$(X : Y) \mapsto (X^p - Y^{p-1}X : Y^p).$$

Taking the parameter $u = Y/X$ around ∞ we find that f is given in that chart by

$$u \mapsto u^p \cdot \frac{1}{1 - u^{p-1}}.$$

Note that ∞ corresponds here to the point $u = 0$ and $\frac{1}{1-u^{p-1}}$ is a unit at 0. One computes, using $p = 0$ in k, that

$$f^* du = -u^{2p-2} \cdot \frac{1}{1 - u^{p-1}} du.$$

Therefore

$$R = (2p-2)[\infty], \qquad \deg(R) = 2p - 2.$$

The Hurwitz formula works now:

$$-2 = 2 \cdot 0 - 2 = p \cdot (2 \cdot -2) + \deg(R) = -2p + (2p - 2).$$

Application 1. *Let $f : X \to Y$ be a surjective (separable) morphism of non-singular projective curves of degree greater than 1. Then*

$$g(X) \geq g(Y).$$

If $g(X) = g(Y)$ then either $g(X) = 0$ or $g(X) = 1$ and in the latter case f is unramified.

Indeed, this follows from analyzing the Hurwitz genus formula. We assume that f is separable since this is the only case we dealt with but in fact the statements hold in general.

Suppose first that $g(Y) \geq 2$. Then

$$2g(X) - 2 = \deg(f)(2g(Y) - 2) + \deg(R) \geq \deg(f)(2g(Y) - 2).$$

As $2g(Y) - 2 > 0$ we get $g(X) > g(Y)$. Similarly, for $g(Y) = 1$ we get $2g(X) - 2 = \deg(R) \geq 0$ and so $g(X) \geq 1 = g(Y)$. In the case $g(Y) = 0$ the inequality is trivial.

We saw that equality can only hold for $g(X) = g(Y) = 0$ or for $g(X) = g(Y) = 1$, and in that case it follows that $\deg(R) = 0$ and so that f is unramified.

Application 2 - Luroth's theorem. *Any subfield of $k(x)$ containing k is a purely transcendental extension of k.*

Let $k(x) \supseteq L \supseteq k$. We may assume $L \neq k$ and thus L is a function field of transcendence degree 1, corresponding to some non-singular projective algebraic curve Y. The inclusion $k(Y) \subset k(x) = k(\mathbb{P}^1)$ corresponds to a surjective morphism $\mathbb{P}^1 \to Y$. Thus, $g(\mathbb{P}^1) \geq g(Y)$. That is, Y has genus 0. It is therefore isomorphic to $\mathbb{P}^1$ (over k) and so $L = k(Y) \cong k(\mathbb{P}^1_k)$ is a purely transcendental extension of k.

Application 3 - hyperelliptic curves. Assume that the characteristic of k is not 2 and let

$$X^0 : y^2 = f(x),$$

be the affine curve in $\mathbb{A}^2_k$, where $f(x)$ is a separable polynomial in $k[x]$ of degree r. Note that X^0 is non-singular as $(2y, f'(x)) = (0, 0)$ at x_0 implies that x_0 is a double root of f, contrary to our assumption. Let X be a non-singular projective model of X^0. Thus, $X^0 \subset X$.

The morphism

$$\varphi : X^0 \to \mathbb{A}^1, \quad \varphi((x, y)) = x,$$

is a surjective morphism of degree 2. It extends to a surjective morphism

$$\varphi : X \to \mathbb{P}^1.$$

Note that if $f(\alpha) \neq 0$ then $(\alpha, \pm\sqrt{f(\alpha)})$ are two distinct points of X^0 and so every pre-image of α lies in X^0 and the map is unramified there. If $f(\alpha) = 0$ then $f(x) = (x - \alpha)g(x)$ and g is a unit of the local ring of $(\alpha, 0) \in X^0$. As $y^2 = f(x)$, we see that y is a uniformizer at $(\alpha, 0)$ and $f(x)$, and so $x - \alpha$, vanish to order 2 there. Thus, φ is ramified at $(\alpha, 0)$. It follows that all the pre-images of α lie also in this case in X^0 and φ has a unique pre-image $P_\alpha := (\alpha, 0)$ of α and $e_{P_\alpha} = 2$. It follows also that $\varphi^{-1}(\infty) = X - X^0$ and $X - X^0$ consists of 1 or 2 points.

Hurwitz formula implies in general that for tamely ramified morphisms $f : X \to Y$, $\sum_{P \in X}(e_P - 1)$ is even. As in our case $e_P = 1, 2$, we conclude that the number of ramification points is even and the r roots of f are among them. Therefore, if f has even degree $r = 2g+2$, we must have $\varphi^{-1}(\infty)$ consisting of two points, both unramified, and the Hurwitz genus formula gives $g(X) = g$. If f has odd degree $r = 2g + 1$ then there is one point at infinity and it is a ramification point. We obtain in this case $g(X) = g$. In particular, we always have $g(X) = \lfloor \frac{\deg(f)-1}{2} \rfloor$.

6.10. **More about topology and complex curves.** To be added (hopefully!)

- { compact Riemann surfaces } $\leftrightarrow$ { projective non-singular curves over $\mathbb{C}$}
- Using topology to construct coverings of a projective non-singular curve over $\mathbb{C}$.
- Belyi's theorem.

7. Morphisms to projective spaces

7.1. **Graded rings, ideals etc.** Let Γ be an abelian group. A Γ-graded ring S is a commutative ring with 1 together with a decomposition into abelian groups,

$$S = \oplus_{\gamma\in\Gamma} S_\gamma,$$

such that for all γ_1, γ_2 in Γ, we have

$$S_{\gamma_1} S_{\gamma_2} \subseteq S_{\gamma_1+\gamma_2}.$$

By a graded ring S we will mean a $\mathbb{Z}$-graded ring S such that $S_n = 0$ for $n < 0$. Thus, $S = \oplus_{n\geq 0} S_n$.

Note: S_0 is a subring and S is an S_0-algebra.

An element of S is homogenous if it belongs to some S_γ. An ideal $\mathfrak{a}\lhd S$ is homogenous if, letting $\mathfrak{a}_\gamma = \mathfrak{a}\cap S_\gamma$, we have

$$\mathfrak{a} = \oplus_{\gamma\in\Gamma}\mathfrak{a}_\gamma.$$

This is equivalent to requiring that $\mathfrak{a}$ is generated by homogenous elements. One checks that a homogenous ideal $\mathfrak{a}$ is prime if and only if for every two homogenous elements f, g of S if $fg \in \mathfrak{a}$ then either $f\in\mathfrak{a}$ or $g\in\mathfrak{a}$.

If $\mathfrak{a}$ is a homogenous ideal then $S/\mathfrak{a} = \oplus_{\gamma\in\Gamma} S_\gamma/\mathfrak{a}_\gamma$ is naturally a Γ-graded ring. Let T be a multiplicative set of homogenous elements of S. Then the localization $S[T^{-1}]$ is also a Γ-graded ring, where

$$S[T^{-1}]_\gamma = \bigcup_{\delta\in\Gamma}\left\{\frac{f}{g} : f\in S_{\gamma+\delta}, g\in S_\delta\right\}.$$

(One needs to check that this is well defined under $\frac{f}{g} = \frac{f_1}{g_1}$.)

The basic example of a graded ring is of course the ring of polynomials $S = A[x_0, \dots, x_n]$ over a ring A. It has a natural grading where $S_0 = A$ and S_d is the A-linear span of the monomials of degree d.

7.2. Proj S. Similar to the case of rings A to which we associated an affine scheme Spec A, we associate here a scheme Proj S to a graded ring S. The construction has some built-in functoriality, but it is not as perfect as in the affine case and, as we shall see, determining morphisms to Proj S is a more involved business.

Let S be a graded ring. The homogenous ideal

$$S_+ = \oplus_{d>0} S_d$$

is sometimes called the irrelevant ideal. Define first Proj S as a set:

$$\text{Proj } S = \{\mathfrak{p}\lhd S : \mathfrak{p} \text{ a homogenous prime ideal and } \mathfrak{p}\not\supseteq S_+\}.$$

Next, we define a topology on Proj S. Let $\mathfrak{a}$ be a homogenous ideal of S and let

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Proj } S : \mathfrak{p} \supseteq \mathfrak{a}\}.$$

Note: We get the same collection of sets $V(\mathfrak{a})$ if we let $\mathfrak{a}$ be any ideal. Indeed, $V(\mathfrak{a}) = V(\mathfrak{a}')$, where $\mathfrak{a}'$ is the homogenous ideal generated by the homogenous elements f of S such that f is a homogenous "part" of some element of $\mathfrak{a}$. That much is clear from the definition. Moreover, we can take $\mathfrak{a}$ to simply be any set of elements of S, as then $V(\mathfrak{a}) = V(\langle\mathfrak{a}\rangle) = V(\langle\mathfrak{a}\rangle')$. This is sometimes convenient.

A calculation similar to the affine case shows that the sets $V(\mathfrak{a})$ satisfy the axioms for closed sets of a topology. For f a homogenous element of S let $D_+(f)$, or $D_+(f;S)$ if needed, be the following set

$$D_+(f) = \{\mathfrak{p} \in \text{Proj } S : f \notin \mathfrak{p}\}.$$

As the complement of $D_+(f)$ is $V(\langle f\rangle)$, $D_+(f)$ is an open set. In fact, it is easy to check that the sets $D_+(f)$, as f ranges over all homogenous elements of S, are a basis for the topology on Proj S.

In order to define a sheaf on Proj S we need the following construction. Let $\mathfrak{p}$ be a prime ideal; let

$$S^0_{(\mathfrak{p})},$$

denote the subring of degree 0 elements of the $\mathbb{Z}$-graded ring obtained by localizing S at all *homogenous* elements of $S - \mathfrak{p}$ (our notation differs a little from Hartshorne's here). This is local ring, whose maximal ideal is generated by elements of the form a/t where a, t are homogenous elements of the same degree, $a \in \mathfrak{p}$, $t \notin \mathfrak{p}$.

Similarly, let h be a homogenous element of S and let S^0_h denote the subring of degree 0 elements of the localization of S at the element h. [9]

Let us now define the sheaf of rings $\mathcal{O}$ on Proj S. Let $U \subseteq$ Proj S be an open set. Let

$$\mathcal{O}(U) = \{f : U \to \coprod_{\mathfrak{p}\in U} S^0_{(\mathfrak{p})} : f(\mathfrak{p}) \in S^0_{(\mathfrak{p})} \text{ and } f \text{ is ``locally-global''}\},$$

where by "locally-global" we mean the following: for all $\mathfrak{p} \in U$ there is an open set $V \subseteq U$ such that $\mathfrak{p} \in V$ and there are elements $a, b \in S_n$ for some n, such that $f = a/b$, viewed in $S^0_{(\mathfrak{q})}$, for all $\mathfrak{q} \in V$ (and, in particular, $b \notin \mathfrak{q}, \forall \mathfrak{q} \in V$). The verification that this is a sheaf of rings is not hard and goes very similarly to the affine case. The sheaf $\mathcal{O}$ has the following properties (see [H] II, 2.5):

(1) $\mathcal{O}_\mathfrak{p} \cong S^0_{(\mathfrak{p})}$;
(2) $(D_+(h), \mathcal{O}_{D_+(h)}) \cong \text{Spec } S^0_h$;
(3) Proj S is a scheme.

[9] We would have to refer in times to the localization of S at h itself and not just to its degree 0 elements. We will denote this localization S_h. This is bad notation as S_h means also the elements of degree h in S. Hopefully, the correct interpretation can be understood from the context.

Example 7.2.1. Let A be a ring and let

$$S = A[x_0, \dots, x_n]$$

be the polynomial ring in $n+1$-variables over A. S is naturally graded (so that $S_0 = A, S_1 = \oplus_{i=0}^n Ax_i$ and so on). We have

$$\text{Proj } S = \cup_{i=0}^n D_+(x_i),$$

as the complement of the union consists of homogenous prime ideals $\mathfrak{p}$ that contain x_i for all i, and hence contain S_+, but those were excluded from Proj S. Further,

$$(D_+(x_i), \mathcal{O}_{D_+(x_i)}) \cong \text{Spec}(A[x_0, \dots, x_n][x_i^{-1}]^0) = \text{Spec } A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right].$$

We therefore see that

$$\text{Proj } S \cong \mathbb{P}^n_{\text{Spec}A},$$

as previously defined. We usually denoted the right-hand side $\mathbb{P}^n_A$ and we shall continue to do so.

7.3. **Morphisms.** Let

$$\varphi : S \to T$$

be a graded homomorphism of graded rings. That is, φ is a ring homomorphism with the additional property $\varphi(S_d) \subseteq T_d$ for all $d \geq 0$. Let

$$U = \{\mathfrak{p} \in \text{Proj } T : \mathfrak{p} \not\supseteq \varphi(S_+)\} = \{\mathfrak{p} \in \text{Proj } T : \varphi^{-1}(\mathfrak{p}) \not\supseteq S_+\}.$$

The set U is an open set as its complement is $V(\varphi(S_+))$; as such it inherits a subscheme structure from Proj T. We claim that there is a natural morphism

$$\begin{array}{l} \text{Proj } T \\ \ \ \mid\!\bigcup \\ \ \ U \xrightarrow{f} \text{Proj } S, \end{array}$$

given on sets by

$$\mathfrak{p} \mapsto f(\mathfrak{p}) := \varphi^{-1}(\mathfrak{p}).$$

The following formulas are easy to check:

- for $\mathfrak{a} \lhd S$ homogenous ideal, $f^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a}))$;
- for $g \in S$ homogenous, $f^{-1}(D_+(g, \text{Proj } S)) = D_+(\varphi(g), \text{Proj } T) \cap U$.

In particular, f is continuous.

The definition of the map on the level of sheaves is done like in the affine case. Let $V \subseteq \text{Proj } S$ be an open set and $g \in \mathcal{O}(V)$. Define f^*g on $f^{-1}(V)$ as follows: for $\mathfrak{q} \in f^{-1}(V)$ let $\mathfrak{p} = f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$. We have a natural ring homomorphism

$$\varphi_{(\mathfrak{a})} : S^0_{(\mathfrak{p})} \to T^0_{(\mathfrak{q})}.$$

Then, we define

$$f^*g(\mathfrak{q}) = \varphi_{(\mathfrak{q})}(g(f(\mathfrak{q}))) = \varphi_{(\mathfrak{q})}(g(\mathfrak{p})).$$

Of course one needs to check that this works, but I leave that to you. On the level of stalks the induced maps that we get are as expected:

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{Proj}\ S,\mathfrak{p}} & \xrightarrow{\varphi_{(\mathfrak{q})}} & \mathcal{O}_{U,\mathfrak{q}} \\ \| & & \| \\ S^0_{(\mathfrak{p})} & & T^0_{(\mathfrak{q})} \end{array}$$

We now put additional assumptions on the graded homomorphism φ. Suppose that for some $d > 0$ the radical of the ideal of T generated by T_d is T_+. This happens, for example, whenever T is a quotient of a polynomial ring $A[x_0 \dots, x_n]$. Assume further that $\varphi : S \to T$ is surjective in degree d. In this case $U = \mathrm{Proj}\ T$ (prime ideals not in U contains $\varphi(S_+)$, hence $\varphi(S_d) = T_d$ and so contain T_+), and so we get a morphism

$$f : \mathrm{Proj}\ T \to \mathrm{Proj}\ S.$$

Now, suppose that φ is surjective; even without the assumption above we find that $U = \mathrm{Proj}\ T$. We claim that f is a closed immersion. Indeed, let $h \in S$ homogenous. Then $f^{-1}(D_+(h, \mathrm{Proj}\ S)) = D_+(\varphi(h), \mathrm{Proj}\ T)$; both source and target are affine and the induced morphism

$$\mathrm{Spec}\ T^0_{\varphi(h)} \to \mathrm{Spec}\ S^0_h,$$

is induced by the surjective ring homomorphism

$$\varphi : S^0_h \to T^0_{\varphi(h)}.$$

It is a closed immersion. In fact, we can be more precise: let $\mathfrak{a} = \mathrm{Ker}(\varphi)$. Then the kernel of $\varphi : S^0_h \to T^0_{\varphi(h)}$ is $\mathfrak{a}[h^{-1}]^0$, the degree zero elements in the localization of $\mathfrak{a}$ at h. Moreover, f is globally a closed immersion, that is, it is also injective onto its image, as our considerations were done using an open cover of $\mathrm{Proj}\ T$.

One easy way to arrive at the setting we discuss here is to take $T = S/\mathfrak{a}$, where $\mathfrak{a}$ is a homogenous ideal. This gives us a closed immersion

$$\mathrm{Proj}(S/\mathfrak{a}) \to \mathrm{Proj}(S).$$

One can show that every closed subscheme of Proj(S) arises this way ([H] II 5.16). But, careful, different ideals $\mathfrak{a}$ may define the same closed subscheme (cf. [H] II, Exe. 3.12).

Example 7.3.1. (The d-uple embedding). Let $d > 0$ an integer. Let $S = \oplus_{n=0}^{\infty} S_n$ and let $S^{(d)} = \oplus_{n=0}^{\infty} S_{dn}$. There is a natural graded homomorphism

$$S^{(d)} \to S.$$

Using considerations similar to those above, as the radical of $S_+^{(d)} = S_+$, we get a morphism

$$f : \text{Proj } S \to \text{Proj } S^{(d)}.$$

This morphism is an isomorphism. In fact, this will follow from the argument about it being an isomorphism on a cover of the image and the pre images of it, but it is instructive to see why it's a bijection. First we check that it is injective. Let $f(\mathfrak{q}) = f(\mathfrak{p})$ then $\mathfrak{q}_{dn} = \mathfrak{p}_{dn}$ for all $n \geq 0$. Let $g \in \mathfrak{q}_n$ then $g^d \in \mathfrak{q}_{dn} = \mathfrak{p}_{dn}$ and so $g \in \mathfrak{p}$, and vice-versa. As $\mathfrak{q}, \mathfrak{p}$ are generated by homogenous elements $\mathfrak{q} = \mathfrak{p}$. Next, we show f is surjective. Let $\mathfrak{b}$ be a homogenous prime ideal of $S^{(d)}$. Let $\mathfrak{a}$ be the radical of the ideal $\langle \mathfrak{b} \rangle$ of S generated by $\mathfrak{b}$. We first claim, that $\langle \mathfrak{b} \rangle_{dn} = \mathfrak{b}_{dn}$. This follows by using $\langle \mathfrak{b} \rangle$ is the abelian group generated by the products $(\oplus S_a)(\oplus \mathfrak{b}_{dn})$. We next show that the same hold for the radical. Indeed, suppose that some homogenous element f of $\mathfrak{a}$ belongs to S_{dn}. Then, for some g, we have $f^g \in S_{dgn} \cap \langle \mathfrak{b} \rangle = \mathfrak{b}_{gdn}$ and so f belongs to the radical of $\mathfrak{b}$ in $S^{(d)}$, which is $\mathfrak{b}$ itself, as it is a prime ideal.

Now, let $h \in S^{(d)}$. We observe that f induces a morphism $D_+(h, S^{(d)}) \to D_+(h, S)$. Moreover, the sheaves of functions are respectively $(S_h^{(d)})^0$ and S_h^0. We claim that these rings are the same. As localization is exact, we have an inclusion $(S_h^{(d)})^0 \to S_h^0$. Let g/h^n be an element of degree zero in S_h^0. Then this forces g to live in degree dn and so $g/h^n \in (S_h^{(d)})^0$.

Now let $S = A[x_0, \dots, x_n]$ with the natural grading and denote by $\{M_\alpha : 0 \leq \alpha \leq \binom{d+n}{n} - 1\}$ the monomials of degree d in the $n+1$ variables x_i. We have established an isomorphism:

$$\mathbb{P}_A^n \cong \text{Proj } S \cong \text{Proj } S^{(d)}.$$

We have $S^{(d)} = A[\{M_\alpha\}]$ where we now give each M_α weight 1. Let $N = \binom{d+n}{n} - 1$. We thus have a graded surjective ring homomorphism

$$A[y_0, \dots, y_N] \to S^{(d)}, \qquad y_i \mapsto M_i.$$

Thus, we get a closed immersion

$$\mathbb{P}_A^n \cong \text{Proj } S^{(d)} \hookrightarrow \mathbb{P}_A^N.$$

This is precisely the <u>d-uple embedding</u> that we had already seen.

Remark 7.3.2. Consider the case of a field k and the structure of

$$S^{(2)} = k[x_0^2, x_0x_1, x_1^2]$$

for $n = 1$. This ring is isomorphic to $k[y_0, y_1, y_2]/(y_1^2 - y_0y_2)$. This ring is not isomorphic to $S = k[x_0, x_1]$ as the affine variety it defines is a cone with singularity at the origin, while S defines the affine variety $\mathbb{A}_k^2$ that is non-singular. We thus conclude that if $X = \text{Proj } S$, the isomorphism class of the homogenous coordinate ring S is not determined by X, unlike in the affine situation.

7.4. **Quasi-coherent shaves on** Proj S. The purpose here is to associated to certain modules M over a graded ring S quasi-coherent sheaves $\tilde{M}$ on Proj S, in similarity to the affine case.

Let $S = \oplus_n S_n$ be a $\mathbb{Z}$-graded ring and let M be a graded S-module. That is, M is an S-module equipped with a decomposition into abelian groups

$$M = \oplus_{n\in\mathbb{Z}} M_n,$$

such that for all d, n we have

$$S_d M_n \subseteq M_{d+n}.$$

If $T \subseteq S$ is a multiplicative set of homogenous elements, $M[T^{-1}]$ is an $S[T^{-1}]$ module. It is a graded module, where elements of the form $\frac{m}{t}$, with $m \in M_n, t \in S_d \cap T$ are given degree $n - d$. One needs to check that this is actually well-defined, etc., but we don't do that here. Our key example is the following:

Fix $d \in \mathbb{Z}$. Let S be a graded ring and let $M = S$ as an S-module but with new "artificial" grading. By definition: $M_n = S_{d+n}$. We denote this graded module $S(d)$. Thus,

$$S(d)_n = S_{d+n}.$$

In particular, $S(d)_0 = S_d$. Note that $S(0) = S$; note too that for $d > 0$, $S(d)$ has elements in negative degree. For example $S(d)_{-d} = S_0$.

To a graded S-module M, as above, we associate a quasi-coherent sheaf $\tilde{M}$ on Proj S in the following way. First, for a homogenous prime ideal $\mathfrak{p}$ denote $M^0_{(\mathfrak{p})}$ the elements of degree 0 in the graded module $M[T^{-1}]$, where T is the set of homogenous elements of $S - \mathfrak{p}$. For an open set $U \subseteq$ Proj S, let

$$\tilde{M}(U) = \{f : U \to \coprod_{\mathfrak{p}\in U} M^0_{(\mathfrak{p})} : f(\mathfrak{p}) \in M^0_{(\mathfrak{p})}, \text{ and } f \text{ is "locally-global"}\}$$

(in the sense we know well by know). Then $\tilde{M}$ is a sheaf and has the following properties ([H] pp. 116-117):

- The stalk $\tilde{M}_\mathfrak{p}$ of the sheaf $\tilde{M}$ at the point $\mathfrak{p}$ is $M^0_{(\mathfrak{p})}$.
- Let $h \in S$ be a homogenous element and M^0_h denote the degree zero elements in the localization $M[h^{-1}]$. Note that M^0_h is an S^0_h-module. Then $\tilde{M}|_{D_+(h)} \cong \widetilde{M^0_h}$.
- In particular, $\tilde{M}$ is a quasi-coherent sheaf.

Notation: the sheaf $\mathcal{O}(n)$ is defined as the sheaf $\widetilde{S(n)}$. In particular, $\mathcal{O} := \mathcal{O}(0)$ is the structure sheaf (more properly denoted $\mathcal{O}_{\text{Proj } S}$).

Let's consider a particular case. Let k be a ring and $S = k[x_0, \dots, x_n]$ with its usual grading. We want to get a precise understanding of the sheaf $\mathcal{O}(m)$ on Proj S. First,

$$\mathbb{P}^n_k = \text{Proj } S = \bigcup_{i=0}^{n} D_+(x_i).$$

By the results above, $\mathcal{O}(m)|_{D_+(x_i)}$ is the sheaf associated to the $k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_0}]$-module which is the degree 0 elements in $(\oplus k[x_0, \dots, x_n]_a)[x_i^{-1}]$, but where now $k[x_0, \dots, x_n]_a$ is considered as having degree $a - m$. That is, this module is

$$\begin{aligned}\sum_{a \geq m} \frac{1}{x_i^{a-m}} k[x_0, \dots, x_n]_a &= x_i^m \sum_{a \geq m} \frac{1}{x_i^a} k[x_0, \dots, x_n]_a \\ &= x_i^m k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \\ &\xrightarrow{\cong} k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}],\end{aligned}$$

where the last isomorphism is an isomorphism of $k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$-modules given by multiplication by x_i^{-m}. To remove possible confusion, we also remark that the equality preceding it holds if the summation is over $a \geq N$ for any N.

Assume now that k is a field. We know that $\mathcal{O}(m)$ is an invertible sheaf on $\mathbb{P}^n_k$, as we have just shown that on the standard affine cover of $\mathbb{P}^n_k$ is it is isomorphic to the structure sheaf. It thus corresponds to a divisor on $\mathbb{P}^n_k$. We will soon prove, in greater generality, that $\mathcal{O}(m_1) \otimes_{\mathcal{O}} \mathcal{O}(m_2) \cong \mathcal{O}(m_1 + m_2)$. Thus, to find the said divisor, it will be enough to consider $\mathcal{O}(1)$. And for that we need a rational section of $\mathcal{O}(1)$.

By the above, x_n is a section of $\mathcal{O}(1)|_{D_+(x_n)}$ that corresponds to the constant function 1 under the isomorphism $\mathcal{O}(1)|_{D_+(x_n)} \cong k[\frac{x_0}{x_n}, \dots, \frac{x_{n-1}}{x_n}]$ given above. Thus, x_n is a non-vanishing section of $\mathcal{O}(1)$ on $D_+(x_n)$. In a natural way x_n can be viewed as a degree 0 element of $S(1)$ that when we restrict to $D_+(x_i)$ gives, under the isomorphism $\mathcal{O}(1)|_{D_+(x_i)} \cong k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$, the function $\frac{x_n}{x_i} \in k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$. We therefore see that

$$(x_n) = H,$$

where H is the hyperplane $x_n = 0$. In particular, under the isomorphism $\deg : \text{Pic}(\mathbb{P}^n_k) \to \mathbb{Z}$, we have

$$\deg(\mathcal{O}(m)) = m.$$

Proposition 7.4.1. *Let S be a graded ring, finitely generated over S_0 by elements of S_1.*

(1) *$\mathcal{O}(m)$ is an invertible sheaf on* Proj S.

(2) *$\mathcal{O}(m_1) \otimes_{\mathcal{O}} \mathcal{O}(m_2) \cong \mathcal{O}(m_1 + m_2)$.*

(3) $\Gamma(\text{Proj } S, \mathcal{O}(m)) = \begin{cases} S_m & m \geq 0, \\ 0 & m < 0. \end{cases}$

Proof. The first part is very similar to the calculation we have just done for the projective space.

As S is generated by S_1,

$$\mathrm{Proj}\, S = \bigcup_{h \in S_1} D_+(h).$$

To show part (1), it thus suffices to prove that $\mathcal{O}(m)|_{D_+(h)} \cong S_h^0$ for every $h \in S_1$. But $\mathcal{O}(m)|_{D_+(h)}$ is the sheaf associated to the module

$$S(m)_h^0 = \sum_{d \geq 0} \frac{1}{h^d} S(m)_d = \sum_{d \geq 0} \frac{1}{h^d} S_{m+d} = h^m \sum_{d \geq 0} \frac{1}{h^{m+d}} S_{m+d} = h^m S_h^0 \cong S_h^0,$$

where the last isomorphism is as S_h^0-modules and is given by multiplication by h^{-m}. In particular, $\mathcal{O}(m)$ is locally-free.

Now, suppose that S is generated over S_0 by the elements $h_0, \dots, h_n$ of S_1. We have

$$\varphi_i : \mathcal{O}(m)|_{D_+(h_i)} \to \mathcal{O}_{D_+(h_i)}, \qquad \varphi_i = h_i^{-m} \cdot (\ ). \tag{7}$$

Therefore, $\mathcal{O}(m)$ corresponds to the 1-cocycle whose value on $D_+(h_i) \cap D_+(h_j)$, for $i < j$, is $(\varphi_j \circ \varphi_i^{-1})$. Namely, $\mathcal{O}(m)$ is defined by the cocycle

$$\xi_{ij} = \left(\frac{h_i}{h_j}\right)^m, \quad i < j.$$

These cocycles depend on m multiplicatively, and so it follows that

$$\mathcal{O}(m_1) \otimes \mathcal{O}(m_2) \cong \mathcal{O}(m_1 + m_2).$$

For the last claim, note first that

$$\Gamma(D_+(h), \mathcal{O}(m)) = h^m S_h^0.$$

Given an element $t \in S_m, m \geq 0$ we therefore get a section of $\mathcal{O}(m)$ on each $D_+(h)$ by writing $t = h^m \cdot \frac{t}{h^m}$; the section corresponds to $\frac{t}{h^m}$ under the trivialization in (7) (and is thus a regular section over each $D_+(h)$). These sections glue as $\xi_{ij} \frac{t}{h_i^m} = \frac{t}{h_j^m}$.

We need to show that these are the only sections of $\mathcal{O}(m)$. To simplify we assume here that $S = S_0[h_0, \dots, h_n]$ is a polynomial ring in $n+1$-variables over S_0, but the general case is very similar ([H] II 5.15). We observe that to give a global section of $\mathcal{O}(m)$ is equivalent to giving $(s_0, \dots, s_n), s_i \in h_i^m S_0[\frac{h_0}{h_i}, \dots, \frac{h_n}{h_i}]$ that are compatible on $D_+(h_i) \cap D_+(h_j) = D_+(h_i h_j)$. Namely, if view the s_i in the $\mathbb{Z}$-graded ring $S' = S[(h_0 \cdots h_n)^{-1}]$ then they all give the same element g and that element is homogenous of degree m.

Now, any homogenous element g of degree m of S' can be written uniquely as

$$g = h_0^{i_0} \cdots h_n^{i_n} f(h_0, \dots, h_n),$$

where f is a homogenous element of S, $h_i \nmid f$ for any i, $i_j \in \mathbb{Z}$ and $\sum_{j=0}^n i_j + \deg(f) = m$. If $g \in S_{h_i}$, then for all $j \neq i$ we must have $i_j \geq 0$. In our case, that holds for every i. Thus, all $i_j \geq 0$. If $m < 0$ that contradicts $\sum_{j=0}^n i_j + \deg(f) = m$; if $m \geq 0$ then $g \in S_m$. $\square$

7.5. **Morphisms to projective space.**

Let k be an algebraically closed field and let X be a quasi-projective variety over k. Let

$$\varphi : X \to \mathbb{P}^n_k = \text{Proj } k[x_0, \ldots, x_n]$$

be a morphism. This produces for us the following data:

- An invertible sheaf $\mathscr{F} := \varphi^* \mathcal{O}(1)$.
- Sections $s_0, s_1, \ldots, s_n$ in $\Gamma(X, \mathscr{F})$ obtained as $s_i = \varphi_i^*(x_i)$.

The s_i have a special property. In general, we say that a sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules is generated by global sections if there is a family $\{s_i \in \Gamma(X, \mathscr{F}) : i \in I\}$ such that for all $P \in X$ a closed point, the sections $\{s_i\}$ viewed in $\mathscr{F}_P$ generate it as an $\mathcal{O}_X$-module. And we say then that $\mathscr{F}$ is generated by $\{s_i : i \in I\}$. Suppose that $\mathscr{F}$ is a coherent $\mathcal{O}_X$-module; using Nakayama's lemma, we have that $\{s_i\}$ generate $\mathscr{F}$ if and only if $\{s_i \pmod{\mathfrak{m}_P}\}$ generate $\mathscr{F}_P / \mathfrak{m}_P \mathscr{F}_P$ over $k = \mathcal{O}_{X,P}/\mathfrak{m}_P$.

In our initial example, the sheaf $\mathcal{O}(1)$ is generated by $\{x_0, \ldots, x_n\}$ and, consequently, the sheaf $\mathscr{F}$ is generated by $\{s_0, \ldots, s_n\}$.

Conversely, *suppose that $\mathscr{F}$ is an invertible sheaf on X generated by global sections $\{s_0, \ldots, s_n\}$. Then, there is a unique morphism*

$$\varphi : X \to \mathbb{P}^n_k,$$

such that $\varphi^* \mathcal{O}(1) \cong \mathscr{F}$ and, under this isomorphism, $s_i = \varphi^* x_i$.

The idea of the proof is simple. Let $X_i = \{P \in X : s_i \notin \mathfrak{m}_P \mathscr{F}_P\}$, which is an open set of X (check on affine cover trivializing $\mathscr{F}$). Then

$$X = \bigcup_{i=0}^{n} X_i.$$

Let U_i be the open set of $\mathbb{P}^n_k$ where $x_i \neq 0$, i.e. $D_+(x_i)$. then $U_i = \text{Spec } k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$. As U_i is affine,

$$\text{Mor}_k(X_i, U_i) = \text{Hom}_k\left(k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}], \Gamma(X_i, \mathcal{O})\right).$$

We take the morphism sending

$$\frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}.$$

(Any two sections of an invertible sheaf differ by a function.) Note that on U_i, $\mathscr{F}$ is trivialized by s_i: under the morphism $X_i \to U_i$ we have constructed, $\mathcal{O}(1)|_{U_i} = k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \cdot x_i$ pulls back to $\mathcal{O}(X_i) \cdot s_i$, x_i pulls-back to s_i and s_j/s_i is a regular function on $D_+(x_i)$.

At this point, it is not clear what properties does the morphism φ have. One question one may ask is *when is it a closed immersion?* This is answered by the following theorem.

Theorem 7.5.1. *Let X be a projective variety, $\mathscr{L}$ an invertible sheaf on X that is generated by the global sections $\{s_0, \dots, s_n\}$. Let V be the k-vector space spanned by $\{s_i\}$ in $\Gamma(X, \mathscr{L})$. Assume the following:*

(1) The elements of V separate points. That is, for any ordered pair of points $P \neq Q$ of X, there is $s \in V$ such that $s_P \in \mathfrak{m}_P \mathscr{L}_P$ and $s_Q \notin \mathfrak{m}_q \mathscr{L}_Q$.

(2) The elements of V separate tangents. That is, for any point P of X the set $\{s_P : s \in V, s_P \in \mathfrak{m}_P \mathscr{L}_P\}$ is equal to the vector space $\mathfrak{m}_P \mathscr{L}_P / \mathfrak{m}_P^2 \mathscr{L}_P$.

Then the morphism $\varphi : X \to \mathbb{P}^n$ corresponding to $\{s_0, \dots, s_n\}$ is a closed immersion.

Remark 7.5.2. The converse also holds, but we don't prove it here. Cf. [H] II 7.3.

Proof. We know already that $\varphi : X \to \mathbb{P}^n$ is a morphism. As X is proper, φ is a closed map. It is also injective. Indeed, take $P \neq Q$. Choose $s = \sum_{i=0}^n \alpha_i s_i$ such that $s_P \in \mathfrak{m}_P \mathscr{L}_P$ and $s_Q \notin \mathfrak{m}_q \mathscr{L}_Q$. Let H be the hyperplane $\sum_{i=0}^n \alpha_i x_i = 0$. Then,

$$\varphi(P) \in H, \quad \varphi(Q) \notin H.$$

In particular, $\varphi(P) \neq \varphi(Q)$. Thus, φ is a bijective bicontinuous morphism. It remains to show that the morphism $\mathcal{O}_{\mathbb{P}^n} \to \varphi_* \mathcal{O}_X$ is surjective. That can be checked on stalks and so we need to show that for every $P \in X$, the homomorphism of local rings

$$\varphi^* : \mathcal{O}_{\mathbb{P}^n, \varphi(P)} \to \mathcal{O}_{X,P},$$

is surjective. This involves some commutative algebra à la Nakayama plus some rather delicate fact. We would need to use that $\varphi_* \mathcal{O}_X$ is a coherent $\mathcal{O}_{\mathbb{P}^n}$-module ([H] II 5.20). This implies that $\mathcal{O}_{X,P}$ is a finitely generated module over $\mathcal{O}_{\mathbb{P}^n, \varphi(P)}$ and so is $\mathfrak{m}_{X,P}$ by Noetherianity. We also know that the two rings have the same residue field, namely k. Furthermore, we have the following

$$\varphi^* : \mathfrak{m}_{\mathbb{P}^n, \varphi(P)} / \mathfrak{m}^2_{\mathbb{P}^n, \varphi(P)} \to \mathfrak{m}_{X,P} / \mathfrak{m}^2_{X,P} \cong \mathfrak{m}_{X,P} \mathscr{L} / \mathfrak{m}^2_{X,P} \mathscr{L}.$$

The composition is surjective by assumption - it is precisely the vector space spanned by the pull back of the linear forms in the x_i that vanish on $\varphi(P)$, namely by the elements s of V such that $s_P \in \mathfrak{m}_{X,P} \mathscr{L}$. It follows that also the first arrow is surjective. Next, Nakayama's lemma implies that $\mathfrak{m}_{\mathbb{P}^n, \varphi(P)} \to \mathfrak{m}_{X,P}$ is surjective. Given any element of $\mathcal{O}_{X,P}$ we can modify it by a scalar so that it lies in $\mathfrak{m}_{X,P}$ and so we easily conclude that $\mathcal{O}_{\mathbb{P}^n, \varphi(P)} \to \mathcal{O}_{X,P}$ is surjective, which is what was to be shown. $\square$

Reformulation in terms of linear systems. Recall the concept of a linear system. If D is a divisor on X then the linear system $|D|$, called a complete linear system, is the set of effective divisors linearly equivalent to D. This set is isomorphic to $\mathbb{P}(V)$, where V is the vector space $\Gamma(X, \mathcal{O}(D))$. Given a non-zero global section s send it to the divisor $(s) + D$, an effective divisor linearly equivalent to D. And vice-versa. A linear system $\mathfrak{d}$ is a linear subvariety of a complete linear system. A point

$P \in X$ is a base-point of $\mathfrak{d}$ if it lies on every divisor $E \in \mathfrak{d}$. Using this concept we can reformulate Theorem 7.5.1 in a more geometric way:

Let $\mathfrak{d}$ be the linear system associated to V such as in the theorem. It is a linear system of effective divisors linearly equivalent to the divisor (s_0), which is $\varphi^{-1}(H)$, where H is the hyperplane $x_0 = 0$.

$$\mathfrak{d} = \{(s) : s \in V = \mathrm{Span}_k\{s_0, \dots, s_n\}\} = \varphi^*\Gamma(\mathbb{P}^n, \mathcal{O}(1)) = \varphi^*(S_1).$$

Then:

- In general, let $\mathscr{L}$ be an invertible sheaf on X and $V \subseteq \Gamma(X, \mathscr{L})$ a subspace: $\mathscr{L}$ is generated by $V \iff$ the linear system $\mathfrak{d} = \mathbb{P}(V)$ has no base-points.

  Assume that this is the case, $\varphi : X \to \mathbb{P}^n$ a morphism such that $\mathscr{L} = \varphi^*\mathcal{O}(1)$ and V is obtained as described above. Then:
- $\mathscr{L}$ separates points $\iff \forall P \neq Q$ there exists $D \in \mathfrak{d}$ such that $P \in D, Q \notin D$.
- $\mathscr{L}$ separates tangents $\iff \forall P \in X$ and $t \in T_P(X)$ there exists $D \in \mathfrak{d}$ such that $P \in D$ and $t \notin T_P(D)$.

Regarding the last condition: the inclusion $D \to X$ allows us to view $T_P(D)$ as mapping to $T_P(X)$. We also note that the last condition also implies that D is non-singular at P.

Example 7.5.3. The complete linear system of hyperplanes of fixed degree $d > 0$ in $\mathbb{P}^n$ has dimension $N := \binom{d+n}{n} - 1$. As it includes dH for any hyperplane H we see that this system has no base points and it separates points. Using $H + (d-1)H'$ for two hyperplanes H and H' we easily check that it separates tangents. Thus, this linear system defines a closed immersion $\mathbb{P}^n \to \mathbb{P}^N$. But, of course, this is just the d-uple embedding.

Example 7.5.4. We leave providing full details as an exercise. Recall the morphism $\mathbb{P}^1 \to \mathbb{P}^2$ given by $(x : y) \mapsto (x^2y : x^3 : y^3)$. If X, Y, Z are the coordinates on $\mathbb{P}^2$ then the image is the cuspidal curve $C : Y^2Z = X^3$. Let H be the hyperplane of $\mathbb{P}^2$ given by $Z = 0$. Prove that $\varphi^*\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^1}(3)$ and that $\{\varphi^*X, \varphi^*Y, \varphi^*Z\} = \{x^2y, x^3, y^3\}$. Using this, we can check that the linear system $\mathfrak{d} = \mathbb{P}(V)$, where $V = \mathrm{Span}\{x^2y, x^3, y^3\}$ (which is not a complete linear system as xy^2, for example, is missing), has no base points (easy) and separates points (easy, but some computation needed).

On the other hand, $\mathfrak{d}$ does not separate tangents. The criterion in terms of linear systems in the case of a curve is simply that given a point P there should be a divisor in $\mathfrak{d}$ that contains P with multiplicity 1. Consider the point $P = (0, 1)$. The general element of V has the shape $\alpha x^2y + \beta X^3 + \gamma y^3$. To vanish at P one must have $\gamma = 0$. Then all sections vanish to order 2 or higher and so T_PD is two dimensional, where D is the closed scheme defined by the homogenous ideal $\langle \alpha x^2y + \beta x^3 + \gamma y^3 \rangle$. It remains to check that the $\mathfrak{d}$ separates tangents at every other point. This gives the known conclusion that $\mathbb{P}^1 - P \cong C - \{(0, 0, 1)\}$ and gives another perspective as to the failure of φ to be an isomorphism.

We finish this section by defining the notion of very ample and ample.

Definition 7.5.5. Let X be a projective variety. An invertible sheaf $\mathscr{L}$ on X is very ample if there is a morphism $\varphi : X \to \mathbb{P}^n$ such that $L \cong \varphi^*(1)$. A divisor D is called very ample if $\mathcal{O}_X(D)$ is very ample. An invertible sheaf $\mathscr{L}$ is ample if for all $n \gg 0$, $\mathscr{L}^{\otimes n}$ is very ample. A divisor D is called ample if for all $n \gg 0$ nD is very ample. A similar definition applies to line bundles.

A very ample sheaf is ample. This follows from the d-uple embedding.

7.6. **Morphisms of a curve to a projective space.** In this section we take the theory developed in the last section and apply it to projective non-singular curves.

Theorem 7.6.1. *Let D be a divisor on a non-singular projective curve X over an algebraically closed field k. Then:*

1. *$|D|$ has no base points $\Longleftrightarrow \forall P \in X$, $\dim |D-P| = \dim |D| - 1$.*
2. *D is a very ample divisor if and only if for every $P, Q \in X$ (including $P = Q$) we have $\dim |D-P-Q| = \dim |D| - 2$.*

Proof. We have an exact sequence that we have used before, for example in the proof of the Riemann-Roch theorem:

$$0 \to \mathcal{O}_X(D-P) \to \mathcal{O}_X(D) \to i_P(k) \to 0.$$

Taking cohomology we find

$$0 \to \Gamma(X, \mathcal{O}(D-P)) \to \Gamma(X, \mathcal{O}(D)) \to k \to H^1(X, \mathcal{O}(D-P)) \cong H^0(X, \mathcal{O}(K+P-D)), \tag{8}$$

and remark that $|D| = \mathbb{P}(\Gamma(X, \mathcal{O}(D)))$ via sending a rational function $s \in \Gamma(X, \mathcal{O}(D))$ to the divisor $(s) + D$. The exact sequence implies that

$$\dim |D| - \dim |D-P| \leq 1$$

(and, in fact, if $H^0(X, \mathcal{O}(K+P-D)) = \{0\}$ then we have equality - we will return to this point later on). Furthermore, we have equality $\dim |D| = \dim |D-P|$ if and only if the map

$$\alpha : |D-P| \to |D|, \qquad E \mapsto E+P,$$

is surjective (as it is clearly injective). Incidentally, this map makes sense because $D \sim D'$ if and only if $D-P \sim D'-P$. We remark that α is also the map induced from $\Gamma(X, \mathcal{O}(D-P)) \to \Gamma(X, \mathcal{O}(D))$. Assume that α is surjective; that is, assume that $\dim |D| = \dim |D-P|$. Then, just because $\alpha(E) = E+P$ it follows that any divisor in $|D|$ contains P and so $|D|$ has a base point P. Conversely, we see that if P is a base point of $|D|$ then α is surjective and $\dim |D| = \dim |D-P|$. This proves (1).

To show (2) note first:

- D is very ample implies that $|D|$ separates points which implies that $|D|$ has no base points.

- If D satisfies $\dim|D-P-Q| = \dim|D|-2$ then it must be that for all P, $\dim|D-P| = \dim|D|-1$, as the dimension cannot drop by more than 1 after each subtraction. Consequently, using (1), $|D|$ has no base points.

Therefore, what we need to show at this point is that if $|D|$ has no base points then D is very ample, if and only if for P, Q, $\dim|D-P-Q| = \dim|D|-2$.

Suppose that D is very ample and hence that $|D|$ separates points and tangents. Therefore, fixing a point P, for all $Q \neq P$ there is $D' \in |D|$ such that $P \in D', Q \notin D'$. That means that $D'-P \notin |D-P-Q|+Q$. As $D'-P \in |D-P|$ it means that $\dim|D-P-Q| = \dim|D-P|-1 = \dim|D|-2$ (as we assume that (1) holds). This works for $P \neq Q$. Note that the reasoning here can be reversed and $\dim|D-P-Q| = \dim|D|-2$ for $P \neq Q$ implies that $|D|$ separates points.

Let us then look at the condition that D separates tangents. That means that for every point P there is $D' \in |D|$ such that P appears with multiplicity 1 in D. Indeed, the condition that $T_P(D')$ doesn't contain a given non-zero vector of $T_P(X)$ is equivalent, as $T_P(X)$ is one dimensional, to $T_P(D')$ being zero dimensional which is equivalent to P appearing with multiplicity 1 in D'. Now,

$$\begin{aligned}
\exists D' \in |D|, \operatorname{mult}(P, D') = 1 &\iff \exists D'' \in |D-P|, \operatorname{mult}(P, D') = 0 \\
&\iff \exists D'' \in |D-P|, D'' \notin |D-2P|+P \\
&\iff \dim|D-P| = \dim|D-2P|+1 \\
&\iff \dim|D| = \dim|D-2P|+2
\end{aligned}$$

(where the last equivalence is due to the fact that (1) holds). □

Theorem 7.6.1 has a number of interesting and elegant consequences.

Corollary 7.6.2. *Let D be a divisor on a non-singular projective curve X.*

(1) If $\deg(D) \geq 2g$ then $|D|$ has no base points.
(2) If $\deg(D) \geq 2g+1$ then D is very ample.

Proof. To show (1) note that $\deg(K+P-D) < 0$, where K is the canonical divisor. Thus, there is no function f such that $(f) \geq -(K+P-D)$. That is, $H^0(X, \mathcal{O}(K+P-D)) = \{0\}$. As noted after Equation (8) this implies that $\dim|D| = \dim|D-P|+1$ for any point $P \in X$ and so, by the theorem, that $|D|$ has no base points.

For (2) we use Riemann-Roch, or more precisely Corollary 6.7.3, to find that $\dim|D-P| = \dim|D|-1$ and $\dim|D-P-Q| = \dim|D|-2$. Indeed, by that Corollary we have $\dim|D| = \deg(D)+g$ (sic!) and similarly for $D-P$ and $D-P-Q$, making the equalities above trivial consequences. □

Corollary 7.6.3. *Let D be a divisor on a non-singular projective curve X. Then D is ample if and only if $\deg(D) > 0$.*

Proof. Suppose that $\deg(D) > 0$. Then $\deg(N \cdot D) \geq 2g+1$ for all $N \geq 2g+1$, and so $N \cdot D$ is very ample and D is ample.

Conversely, suppose that D is ample then for all $N \gg 0$, $N \cdot D$ is very ample. Let $\varphi : X \to \mathbb{P}^n$ be the closed immersion associated with $N \cdot D$. Then $\mathcal{O}(N \cdot D) \cong \varphi^*\mathcal{O}(1)$. But, under the functoriality of divisors and invertible sheaves, $\mathcal{O}(N \cdot D)$ is isomorphic to $\mathcal{O}(R)$, where R is the divisor obtained by intersecting X with a hyperplane on which X doesn't lie. In particular, R is an effective divisor and so $\deg(D) = N^{-1}\deg(N \cdot D) = N^{-1}\deg(R) > 0$. □

Remark 7.6.4. Thus, $\deg \varphi^*\mathcal{O}(1)$ is called the <u>degree</u> of the curve $\varphi(X)$ in $\mathbb{P}^n$. It is the number of intersection points the curve has with a generic hyperplane in $\mathbb{P}^n$.

Example 7.6.5. Let X be a curve of genus 1 and choose a point $t \in X$. The divisor $D = 3[t]$ has degree $3 = 2 \cdot 1 + 1$ is thus very ample. It defines a closed immersion

$$X \to \mathbb{P}^2,$$

whose image is a curve of degree 3. But, in $\mathbb{P}^2$, Bezout's theorem implies that the image is a cubic in $\mathbb{P}^2$. This is more or less exhibiting X by means of a Weierstrass equation. ("More or less" as a simplification of the cubic by means of change of variables is required first; or one should choose the sections carefully as we had done when deriving the Weierstrass equation.)

Example 7.6.6. Suppose that X is a curve of genus 2. Then every divisor of degree $5 = 2 \cdot 2 + 1$ is very ample. Thus, if D is a divisor of degree 5 on X, $\dim |D| = \deg(D) - g(X) = 3$ and X embeds in $\mathbb{P}^3$ as a curve of degree 5. We remark that X does not embed in $\mathbb{P}^2$ as 2 is not an integer of the form $(d-1)(d-2)/2$.

7.6.1. *The canonical morphism of a curve.*

Theorem 7.6.7. *If X is a non-singular projective curve of genus $g \geq 2$. Then the linear system $|K|$ has no base points and thus defines a morphism $\varphi : X \to \mathbb{P}^{2g-1}$. This morphism is called the <u>canonical morphism</u>. The morphism φ is a closed immersion (that is K is very ample) unless X is hyperelliptic.*

Proof. Using Theorem 7.6.1, we need to show that $\dim |K| = \dim |K - P| + 1$ for any point P. We have, by definition, $\dim |K| = h^0(X, \mathcal{O}_X(K)) - 1 = g - 1$. By Riemann-Roch and Serre's duality, $\dim |K - P| = \deg(K - P) - g + h^0(X, \mathcal{O}_X(P)) = (2g-3) - g + h^0(X, \mathcal{O}_X(P))$. If there is a function with a simple pole at P then it defines an isomorphism $X \to \mathbb{P}^1$ which implies that X has genus 0. Thus, $H^0(X, \mathcal{O}_X(P)) = k$ and $h^0(X, \mathcal{O}_X(P)) = 1$ and we conclude that $\dim |K - P| = g - 2$.

Let

$$\varphi : X \to \mathbb{P}^{2g-1},$$

be the morphism associated to the linear system K.

Now, K is very ample if and only if $\dim |K| = \dim |K - P - Q| + 2$ for all P, Q. As noted $\dim |K| = g - 1$. Furthermore, $\dim |K - P - Q| = \deg(K - P - Q) - g + h^0(X, \mathcal{O}_X(P + Q)) = g - 4 + h^0(X, \mathcal{O}_X(P + Q))$. Thus, $|K|$ is very ample if and only if for any two points P, Q on X we have $h^0(X, \mathcal{O}_X(P+Q)) = 1$. As we have $k \subseteq H^0(X, \mathcal{O}_X(P+Q))$, $|K|$ fails to be very ample if and only if for any two points P, Q on X there is a non-constant function f with poles at most $P+Q$. If f has a single pole, it defines an isomorphism $X \to \mathbb{P}^1$, which is not possible. If f has two poles, then f defines a surjective morphism of degree 2, $X \to \mathbb{P}^1$ which means that X is hyperelliptic. This can be taken as the definition of being hyperelliptic, or, argue that $k(X)$ is a quadratic extension of $k(\mathbb{P}^1)$ thus of the form $k(x)[y]/(y^2 - f(x))$, where f is not a square. We may modify f by squares, and so assume that f is a polynomial with distinct roots, arriving at the other definition of hyperelliptic. □

Remark 7.6.8. Implicit in the proof is that every curve of genus 2 is hyperelliptic. Indeed, if K is very ample, the dimension of the linear system $|K-P-Q| = g-3$, yet must be non-negative. We can also show it more quickly as follows (we do that just for fun). Let X be a curve of genus 2. Then, there is a regular global differential on X and its divisor K is thus effective and of degree 2. Thus, $K = P+Q$ for some points P, Q on X, possibly equal. Consider $H^0(X, \mathcal{O}_X(P+Q))$. Using Riemann-Roch, $h^0(X, \mathcal{O}_X(P+Q)) - h^0(X, \mathcal{O}_X(K-(P+Q))) = 1$. But, $h^0(X, \mathcal{O}_X(K-(P+Q))) = h^0(X, \mathcal{O}_X) = 1$. That is, $h^0(X, \mathcal{O}_X(P+Q)) = 2$. As such, there is a non-constant function in $H^0(X, \mathcal{O}_X(P+Q))$. If it has a single pole, as above, $X \cong \mathbb{P}^1$. Thus, it must have poles at both P, Q (each simple if $P \neq Q$, double pole if $P = Q$). This provide a double cover $X \to \mathbb{P}^1$.

8. Surfaces