EXERCISES FOR ALGEBRAIC GEOMETRY II, WINTER 2015

Solve the following exercises:

(1) Let $A \to B$ be a ring homomorphism. Prove the following.
  (a) Let A_1 be an A-algebra and define $B_1 = A_1 \otimes_A B$, which is an A_1-algbera and a B-algebra. Then
$$\Omega_{B_1/A_1} = B_1 \otimes_B \Omega_{B/A}.$$
  (b) Let S be a multiplicative set in B then
$$\Omega_{B[S^{-1}]/A} = B[S^{-1}] \otimes_B \Omega_{B/A}.$$

(2) Let C be a ring. To show that a complex of C-modules,
$$M_1 \xrightarrow{v} M_2 \xrightarrow{u} M_3 \ ,$$
is exact, it suffices to show that for every C-module T, the following sequence is exact:
$$\mathrm{Hom}_C(M_1, T) \xleftarrow{v^*} \mathrm{Hom}_C(M_2, T) \xleftarrow{u^*} \mathrm{Hom}_C(M_3, T) \ .$$

(3) Let C be a ring. A homomorphism of C-modules $v : M_1 \to M_2$ is injective and its image a direct summand, if and only if the homorphism $v^* : \mathrm{Hom}_C(M_2, T) \to \mathrm{Hom}_C(M_1, T)$ is surjective for all C-modules T.

(4) Let k be a ring and A a k-algebra. Then:
  (a) $\Omega_{A[\underline{x}]/k} = (\Omega_{A/k} \otimes_A A[\underline{x}]) \oplus \oplus_{i=1}^n A[\underline{x}] \cdot dx_i$ (the canonical isomorphism being induced by Proposition 1.2.1).
  (b) Let $\mathfrak{m} = \langle f_1, \dots, f_m \rangle$ be an ideal of $A[\underline{x}]$ and let $C = A[\underline{x}]/\mathfrak{m}$. Show that
$$\Omega_{C/k} \cong (\Omega_{A/k} \otimes_A (A[\underline{x}]/\mathfrak{m})) \oplus \oplus_{i=1}^n (A[\underline{x}]/\mathfrak{m}) \cdot dx_i,$$
  modulo $\delta(\mathfrak{m}/\mathfrak{m}^2)$, where
$$\delta(f) = (d_0 f)(x) + \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i,$$
  and where for $f = \sum_I a_I x^I$, $a_I \in A$ we let
$$(d_0 f)(x) = \sum_I d_{A/k} a_I (\mathrm{mod}\ \mathfrak{m}) \cdot x^I.$$

(5) Let $(A, \mathfrak{m})$ be a local ring. Let M be an A module that is finitely generated. Suppose that $x_1, \dots, x_n$ are elements of M that generate the A-module $M/\mathfrak{m}M$. Then $x_1, \dots, x_n$ generate M.

(6) Do Hartshorne, Chapter I, exercises 5.1.

(7) Do Hartshorne, Chapter I, exercises 5.2.

(8) Do Hartshorne, Chapter I, exercises 5.3.

(9) Do Hartshorne, Chapter I, exercises 5.5.

(10) Given a non-zero polynomial $f \in k[x_1, \dots, x_n]$ write f as a sum of its homogenous parts

$$f = f_r + \cdots + f_N,$$

where f_i is the homogenous part of f of weight i and $f_r \neq 0$. Define

$$f^* := f_r,$$

and define for an ideal I of $k[x_1, \dots, x_n]$,

$$I^* = \langle f^* : f \in I \rangle.$$

Prove that I^* is a homogeneous ideal. Show by example that if $I = \langle f_1, \dots, f_m \rangle$ then $I \supseteq \langle f_1^*, \dots, f_m^* \rangle$, but they may not be equal. Show by example that I need not be a radical ideal.

Show, however, that if $I = \langle f \rangle$ is a principal ideal then $I^* = \langle f^* \rangle$. Calculate I^* for the cuspical and nodal curves.

(11) Let Y be an affine variety over k with coordinate ring $k[Y] = k[x_1, \dots, x_n]/I$. Assume that $\underline{0} \in Y$. Define the tangent cone to Y at $\underline{0}$ as the scheme

$$C_{Y,\underline{0}} = \operatorname{Spec}(k[x_1, \dots, x_n]/I^*).$$

Let us write $k[x_1, \dots, x_n] = \oplus_{a=0}^{\infty} k[x_1, \dots, x_n]_a$, the sum of the homogenous parts. Prove that if $I = \langle f_1, \dots, f_m \rangle$ then $I^* \cap k[x_1, \dots, x_n]_1 = \langle f_{1,1}, \dots, f_{m,1} \rangle$. Deduce that the tangent space T to the tangent cone at $\underline{0}$ is equal to the tangent space $T_{Y,\underline{0}}$ of Y at $\underline{0}$ and that there is a natural closed immersion

$$C_{Y,\underline{0}} \hookrightarrow T_{Y,\underline{0}}.$$

(12) Give an example of a curve Y in $\mathbb{A}^3$, passing through $\underline{0}$, such that $T_{Y,\underline{0}} = \mathbb{A}^3$ and whose tangent cone consists of lines whose linear span is $T_{Y,\underline{0}}$. In contrast give an example of a curve Y in $\mathbb{A}^3$, passing through $\underline{0}$, such that $T_{Y,\underline{0}} = \mathbb{A}^3$ and the reduced underlying scheme of $C_{Y,\underline{0}}$ is a single line.

(13) Let A be a local ring with maximal ideal $\mathfrak{m}$. Define the associated graded ring,

$$gr(A) = \oplus_{a=0}^{\infty} \mathfrak{m}^a/\mathfrak{m}^{a+1},$$

(where, by definition, $\mathfrak{m}^0 = A$). Let $k = A/\mathfrak{m}$ prove that $gr(A)$ is a graded k-algebra. Prove that if $x_1, \dots, x_n$ generate $\mathfrak{m}/\mathfrak{m}^2$ then there is an isomorphism

$$gr(A) \cong k[x_1, \dots, x_n]/I^*,$$

where I^* is some homogenous ideal of $k[x_1, \dots, x_n]$, where the isomorphism is as graded rings.

Suppose next that Y is an affine variety defined by an ideal I and that $\underline{0} \in Y$. Let $A = \mathcal{O}_{Y,\underline{0}}$, with maximal ideal $\mathfrak{m}A$, where $\mathfrak{m} = (x_1, \dots, x_n)/I$. Prove that

$$gr(A) \cong k[x_1, \dots, x_n]/I^*,$$

where I^* is the ideal generated by the leading homogenous terms of the elements of I. Conclude,

$$C_{Y,\underline{0}} \cong \operatorname{Spec}(gr(A)).$$

(14) The Cayley cubic is a singular surface given in $\mathbb{P}^3$ by the equation $\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0$, which we can write in polynomial form by multiplying by $x_0x_1x_2x_3$. Note that there is an action of S_4 on this surface.

Find the singular points of this surface. There are 4 of them. Show that any two singular points lie on a line lying on the surface. This gives 6 lines. Find the tangent cone at each singular point. Prove that there are at least 3 more lines on the Cayley cubic. One of them is given by the equations $x_0 + x_1 = x_2 + x_3 = 0$. In fact, these 9 lines are all the lines lying on the Cayley cubic, but this requires some work. Find the overall configuration of intersections between the 9 lines.

The Cayley cubic is the unique singular cubic in $\mathbb{P}^3$, up to isomorphism, with 4 ordinary double points and no other singular points (4 ordinary double points is in fact the maximal number of ordinary double points possible for a cubic surface).

(15) Do the exercises [H] II.4.5 (a), (b).

(16) This exercise is taken from [AM] Exercises 28 and 32, page 72.

Let Γ be a totally ordered abelian group. A subgroup Δ of Γ is called isolated in Γ if, whenever $0 \leq \beta \leq \alpha$ and $\alpha \in \Delta$ then $\beta \in \Delta$. (Perhaps a better name would have been convex.)

(a) Let A be a valuation ring with fraction field K and value group Γ. Let $\mathfrak{p}$ be a prime ideal of A. Show that $v(A - \mathfrak{p})$ is the set of non-negative elements of an isolated subgroup Δ of Γ. Show further that the mapping so defined of $\operatorname{Spec}(A)$ into the set of isolated subgroups of Γ is bijective. (One defines the rank of the valuation as the length n of a maximal chain of isolated subgroups $\Delta_0 \subsetneq \cdots \subsetneq \Delta_n$. Note that this is therefore just the Krull dimension of A).

(b) Deduce from this correspondence that the set of prime ideals of A is totally ordered.

(c) If $\mathfrak{p}$ is a prime ideal, prove that $A/\mathfrak{p}$ and $A_{\mathfrak{p}}$ are valuation rings as well. What are the value groups for these valuations?

(17) (Example of a valuation ring of rank 2). Consider the abelian group $\mathbb{Z}^2$ with the lexicographic order: $(a, b) < (a', b')$ if either $a < a'$, or $a = a'$ and $b < b'$. Show that this is a linearly ordered abelian group. Find its isolated subgroups.

We now proceed to find a field with a valuation in this group. Let K be the field of formal power series in two variables and complex coefficients satisfying the following restrictions: every element of K is a power series $\sum_{r \geq a}(x^r \sum_{s \geq b(r)} c_{r,s} y^s)$, where a is an integer and $b(r)$ is an integer depending on r.

(a) Show that K is a field.

(b) Given an element of K as above, define its valuation as the minimal (r, s) for which $c_{r,s} \neq 0$.

(c) Find the valuation ring and its prime ideals.

(18) Show that the affine curves given by $y = x^2$ and $xy = 1$ are birational but not isomorphic.

(19) Show further, that for every irreducible quadratic polynomial $f(x,y) \in k[x,y]$ the conic section defined by $f(x,y) = 0$ in $\mathbb{A}^2$ is isomorphic to precisely one of the curves above and give a criterion to determine which. (This is [H] Ex. I 1.1, which is much easier to do once we have all the theory we have developed!)

(20) Show that the group $\mathrm{PGL}_2(k) := \mathrm{GL}_2(k)/k^*$ acts faithfully as automorphisms of $\mathbb{P}^1_k$ via the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} t := \frac{at+b}{ct+d}$$

(Möbius transformations), where we have identified the function field of $\mathbb{P}^1_k$ with that of $\mathbb{A}^1_k = \mathrm{Spec}\, k[t]$. Show further that any automorphism of $\mathbb{P}^1_k$ arises this way. That is

$$\mathrm{Aut}_k(\mathbb{P}^1_k) = \mathrm{PGL}_2(k).$$

(It is also true that $\mathrm{Aut}(\mathbb{P}^n_k) = \mathrm{PGL}_{n+1}(k)$.)

(21) Let $P_1, \ldots, P_a$ be distinct closed points of $\mathbb{A}^1_k$ and $Q_1, \ldots, Q_b$ another distinct set of distinct points of $\mathbb{A}^1_k$. Prove that if $\mathbb{A}^1 - \{P_1, \ldots, P_a\} \cong \mathbb{A}^1 - \{Q_1, \ldots, Q_b\}$ then $a = b$. Show that the converse may fail - what is a minimal counter-example?

(22) Consider the projective curve $C_d : x^d + y^d + z^d = 0$ in $\mathbb{P}^2_k$ (the Fermat curve). Assume k has characteristic 0 (to simplify the calculations). Show that the rational map $(x : y : z) \mapsto (x : y)$ defines a dominant morphism $C_d \longrightarrow \mathbb{P}^1_k$. Calculate the degree of this map. Determine all points $(x : y)$ in which the closed points of the fibre have cardinality smaller than the degree and determine precisely the cardinality of the fibre at those points.

The curve C has a large group of automorphisms. Which of those automorphisms commutes with the morphism $C_d \longrightarrow \mathbb{P}^1$? Is the field extension $k(C_d) \supseteq k(\mathbb{P}^1)$ Galois?

(23) Assume characteristic zero to simplify. Show that the non-singular curve associated to the cuspidal curve $y^2 = x^3$, as well as to the nodal curve $y^2 = x^2(x+1)$, is $\mathbb{P}^1$. In both cases provide a surjective birational morphism from $\mathbb{P}^1$ to the closure of the curve in $\mathbb{P}^2$, namely to $y^2z - x^3 = 0$ and $y^2z - x^2(x+1) = 0$.

(24) Denote by j the inclusion map $j : V \to U$ of an open subset V of a topological space U. We have the operation $j_!$ of extension by zero from sheaves on V to sheaves on U. Namely, if $\mathscr{J}$ is a sheaf on V we define $j_!\mathscr{J}$ to be the sheaf on U associated to the presheaf with the values

$$W \mapsto \begin{cases} 0 & W \not\subseteq V \\ \mathscr{J}(W) & W \subseteq V. \end{cases}$$

Prove that

$$(j_!\mathscr{J})_P = \begin{cases} 0 & P \notin V \\ \mathscr{J}_P & P \in V. \end{cases}$$

and that $j_!\mathscr{J}$ restricted to V is $\mathscr{J}$.

(25) Assume for simplicity that the base field k is algebraically closed of characteristic zero. Calculate the zero-th and first cohomology of the projective non-singular plane curve $C : x^3 + y^3 + z^3 = 0$ for the sheaves $\mathcal{O}, \Omega_{C/k}$, using the affine cover of C induced from the standard affine cover of $\mathbb{P}^2_k$ by three copies of $\mathbb{A}^2_k$ (note that C is in fact covered already by any two of these three open sets, which simplifies the calculations). We provide some

hints: (i) The dimension of all these cohomology groups is 1. (ii) Note that choosing an affine model $s^3+t^3+1=0$, where $s=x/z, t=y/z$ any differential on C can be written as $f(s,t)ds$, with $f(s,t)\in k(C)^\times$. (iii) Show that the differential $\omega := t^{-2}ds = -s^{-2}dt$ is a holomorphic global differential and calculate its divisor. Namely, for every point $P\in C$, choose a local uniformizer at P, say w_P and express this differential in the local ring as $g\cdot dw_P$ and find the valuation of g. (iv) Using this, show that any other non-zero holomorphic differential is a scalar multiple of ω.

(26) Calculate $\check{\mathrm{H}}^1(\mathbb{A}_k^2-\{0\},\mathcal{O}_X)$ using the cover $x\neq 0$ and $y\neq 0$ (that are both affine). Show that it is not zero. More precisely, show that it is isomorphic $\bigoplus_{i,j>0} k\cdot\frac{1}{x^iy^j}$. Using that this Cech cohomology actually calculates $H^1(\mathbb{A}_k^2-\{0\},\mathcal{O}_X)$, and comparing with Theorem 5.4.5, conclude again that $\mathbb{A}_k^2-\{0\}$ is not affine.

(27) Let $f:X\to Y$ be an affine morphism of noetherian separated schemes over k. Let $\mathscr{F}$ be a quasi-coherent sheaf on X. Prove that

$$H^i(X,\mathscr{F})\cong H^i(Y,f_*\mathscr{F}).$$

(On a separated scheme the intersection of affine subsets is affine. For the notion of affine morphisms see [H] II, Exercise 5.17. You may freely use it.) Here are two cases where this exercise applies: (i) The closed immersion of a point of X into X; (ii) Any non-constant morphism between projective, possibly singular, curves (any finite morphism of schemes is affine).

The following sets of exercises concerns the blow up of a variety X at a point. It follows [H] I.4. Many of the properties you are asked to prove can be found there, so I request that you try and prove them without consulting the proofs of [H]. Although, after you had found a proof, there is no harm in comparing notes with [H].

(28) **The blow-up of $\mathbb{A}^n$ at the point 0.** The blow-up of a point on $\mathbb{A}^n$ will be constructed as a closed subset of $\mathbb{A}^n\times\mathbb{P}^{n-1}$. Closed subsets of this product can be described by polynomials in $2n$-variables, $f(x_1,\dots,x_n;y_1,\dots,y_n)$ (note the unusual numbering of the coordinates y_i on $\mathbb{P}^{n-1}$) that are, in addition, homogenous in the variables y_i. That is, expanding in monomials in the y_i's with coefficients in $k[x_1,\dots,x_n]$, all monomials have the same degree.

The blow-up of $\mathbb{A}^n$ at zero is the closed subvariety of $\mathbb{A}^n\times\mathbb{P}^{n-1}$ defined by the following equations:

$$x_iy_j=x_jy_i,\qquad \forall i,j.$$

Show that the projection map $\pi:\mathbb{A}^n\times\mathbb{P}^{n-1}\to\mathbb{A}^n$ is a birational morphism, isomorphism outside $\{0\}\in\mathbb{A}^n$ and $E:=\pi^{-1}(0)=\{0\}\times\mathbb{P}^{n-1}$ (called the exceptional fibre). Show that is a natural bijection between lines ℓ through the origin in $\mathbb{A}^n$ and points on E that has the following property (denote by $\bar{\ }$ Zariski closure): $\overline{\pi^{-1}(\ell-\{0\})}\cap E$ is the point corresponding to ℓ.

In a similar way we define the blow up of $\mathbb{A}^n$ at any closed point. Write down the equations and properties for this blow-up. We write the resulting quasi-projective variety $\mathrm{Bl}_v(\mathbb{A}^n)$.

(29) **The blow-up of an affine variety V at a point $v\in V$.** Now let $V\subseteq\mathbb{A}^n$ be a closed subset and let $v\in V$ be a closed point. For simplicity we assume: (i) V is irreducible and $\dim(V)>0$. (ii) v is the zero point of $\mathbb{A}^n$. We define the blow-up of V at 0, $\mathrm{Bl}_0(V)$ as the Zariski closure of $\pi^{-1}(V-\{v\})$ in $\mathrm{Bl}_0(\mathbb{A}^n)$.

Prove that there is a birational morphism $\mathrm{Bl}_0(V) \to V$, that is zero outside $E_Y := E \cap \mathrm{Bl}_0(V)$ and in particular $\mathrm{Bl}_0(V)$ is irreducible. Let $I(V)$ be the prime ideal defining V. Then $\mathrm{Bl}_0(V)$ is defined by the ideal

$$\{f(\underline{x}; \underline{y}) : f(\underline{x}; \underline{x}) \in I(V)\}$$

(Note that this ideal contains the ideal $\langle \{x_i y_j - x_j y_i : 1 \leq i, j \leq n\} \rangle$.)

(30) **The exceptional divisor.** Continuing the following exercise, we provide a very insightful description of the special fibre of the blow-up of Y, $E_Y := \mathrm{Bl}_0(Y) \cap \{0\} \times \mathbb{P}^{n-1}$.

Let $C_{Y,0}$ be the tangent cone to Y at 0. Let C be the cone in $\{0\} \times \mathbb{A}^n$ that lies over E_Y. Namely, the k-points of C are the zero point and the non-zero vectors $(0, \dots, 0; y_1, \dots, y_n)$ such that $(0, \dots, 0; y_1 : \cdots : y_n) \in E_Y$. Prove that

$$C \cong C_{Y,0}.$$

Another way to put it is that E_Y is the projectivization of the tangent cone to Y at the point of the blow-up.

[Suggestion: consider the homogenous ideal

$$J = \langle \{f(0, \dots, 0; y_1 : \cdots : y_n) : f(x_1, \dots, x_n; x_1, \dots, x_n) \in I(Y)\} \rangle.$$

Show that $Z(J) = E_Y$ and so the statement to be proven is that

$$J = I(Y)^*.$$

I can't say it's straightforward, but try!]

(31) **Examples of Blow-up.** Show that the blow-up at 0 of the affine curves $y^2 = x^3$ and $y^2 = x^2(x+1)$ are non-singular curves and determine their special fibres.

(32) **Another example of Blow-up.** Show that the blow-up of the cone $\{x^2 + y^2 = z^2\} \subseteq \mathbb{A}^3$ is a non-singular surface and determine the special fibre.

(33) **The blow-up of $\mathbb{P}^n$ at a point $p \in \mathbb{P}^n$.** There is a projective version of blow-up. Consider $\mathbb{P}^n \times \mathbb{P}^{n-1}$ with coordinates $x_0, \dots, x_n$ on $\mathbb{P}^n$ and $y_1, \dots, y_n$ on $\mathbb{P}^{n-1}$. Closed subvarieties of this product are described as the zero set of polynomials $f(x_0, x_1, \dots, x_n; y_1, \dots, y_n)$, homogenous in each set of variables separately (so, for example, $x_0^3 y_3 + x_1^2 x_2 y_1$, but not $x_0^2 y_3^2 + x_1^2 x_2 y_1$). Consider the closed subset $X = \mathrm{Bl}_P(\mathbb{P}^n)$ of $\mathbb{P}^n \times \mathbb{P}^{n-1}$ defined by the equations

$$x_i y_j = x_j y_i, \quad i, j = 1, \dots, n.$$

Prove the following. Let $P = (1 : 0 : \cdots : 0) \in \mathbb{P}^n$. The projection morphism $\pi : X \to \mathbb{P}^n$ is an isomorphism outside $\pi^{-1}(P)$. Let $E = \pi^{-1}(P)$ then $E \cong \mathbb{P}^{n-1}$ and there is a natural bijection between points of E and lines ℓ of $\mathbb{P}^n$ passing through P. If ℓ is such a line and p_ℓ the corresponding point then the closure of $\pi^{-1}(\ell - \{P\})$ in X intersects E at a unique point, which is p_ℓ. In addition X is irreducible.

(34) **The blow-up of a projective variety Z at a point $z \in Z$.** Let Z be a closed irreducible subvariety of $\mathbb{P}^n$ passing through P and of positive dimension. (Once more, the more general case is handled by change of coordinates). Define the blow up of Z at P, $\mathrm{Bl}_P(Z)$ as the Zariski closure of $\pi^{-1}(Z - \{P\})$ in $X = \mathrm{Bl}_P(\mathbb{P}^n)$. Show that $\pi : \mathrm{Bl}_P(Z) \to Z$ is a birational morphism that is an isomorphism outside $E_Z = \mathrm{Bl}_P(Z) \cap (\{P\} \times \mathbb{P}^{n-1})$. Show that $\mathrm{Bl}_P(Z)$ is irreducible.

(35) **Reconciling two approaches to blow-up.** Let Z be as in the previous exercise. Consider the standard affine chart around P in which P corresponds to the zero point in $\mathbb{A}^n = \operatorname{Spec} k[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}]$. Show that, under the natural identifications, $\mathrm{Bl}_P(Z) \cap (\mathbb{A}^n \times \mathbb{P}^{n-1}) = \mathrm{Bl}_0(Z \cap \mathbb{A}^n)$.

(36) **An example of Blow-up (projective version).** Let Z be the projective closure in $\mathbb{P}^3$ (with coordinates $x_0, \dots, x_3$) of the affine cone $C : x_1^2 + x_2^2 = x_3^2$ in $\mathbb{A}^3$. Find $T = Z - C$. Calculate the blow-up $\mathrm{Bl}_P(Z)$ of Z at the point P. For every line ℓ in $\mathbb{P}^3$ passing through P, calculate the intersection of ℓ with the exceptional fibre $E_Z = \mathrm{Bl}_P(Z) \cap \{P\} \times \mathbb{P}^2$ of the blow-up and with T. Draw a picture of $\mathrm{Bl}_P(Z)$ and how it relates to Z.

(37) Consider the linear system of cubics passing through a given set $\{P_1, \dots, P_6\}$ of six district points of $\mathbb{P}^2$, no three of which are co-linear. Prove that this linear system is 3-dimensional (which means that the space of such cubics itself is 4-dimensional). Let $\{f_0, \dots, f_3\}$ be a basis for that system and consider the map

$$x \mapsto (f_0(x) : \cdots : f_3(x)).$$

Show that it gives an injective morphism $U := \mathbb{P}^2 - \{P_1, \dots, P_6\} \to \mathbb{P}^3$. This is the first step in showing that the blowup of $\mathbb{P}^2$ at 6 points is a cubic surface in $\mathbb{P}^3$.

(38) The Clebsch cubic discussed in class is isomorphic to the following surface in $\mathbb{P}^4$ (that has an obvious faithful S_5 action):

$$x_0 + x_1 + x_2 + x_3 + x_4 = 0,$$
$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.$$

One line on this surface is given by $x_0 = 0, x_1 + x_2 = 0$. Letting the symmetry group S_5 act, how many lines of this form do we get? For each pair of distinct lines what is their intersection number? Assume the following fact: if D is a divisor on a non-singular projective surface S and F is a principal divisor on S then $D.F = 0$. This implies that for every non-zero function f and any two divisors D_1, D_2 on S, $D_1.D_2 = D_1.(D_2 + (f))$. Use this to calculate the self-intersection of the lines you found above.

Now go back to the model of $\mathbb{P}^2$ blown-up at 6 points. What do you think are the lines above? Can you explain in this model the results about the intersection numbers you found above? (I am asking for a heuristic explanation only).
[Example: Let H be a line on $\mathbb{P}^2$, which is a divisor. Then H is given by a linear form $\ell = 0$. Choose another linear form $m = 0$, not proportional to ℓ. Then m/ℓ is a function on $\mathbb{P}^2$ with divisor $H_1 - H$, where H_1 is the zeros of m. Then, $H.H = H.(H + (H_1 - H)) = H.H_1 = 1$, as two distinct lines intersects at exactly one point.]

(39) Let X be a non-singular projective curve over an algebraically closed field k, and $\{P_1, \dots, P_n\}$ distinct (closed) points of X, $n \geq 1$. Prove that $X - \{P_1, \dots, P_n\}$ is an affine curve.

(40) Let E be a non-singular curve of genus 1 over an algebraically closed field k. Fix a point P_0 on C (By a "point" we mean a closed point). Let P, Q be two points on E, not necessarily distinct. Prove that there is a unique point R on E such that $[P] + [Q] - [R] - [P_0]$ is a

principal divisor. Define a map

$$E(k) \times E(k) \to E(k), \qquad (P, Q) \mapsto R,$$

where P, Q and R are related as above. Prove that this addition law makes $E(k)$ into an abelian group whose identity element is P_0. It is less easy to show that in fact there is a morphism $E \times E \to E$ making E into a group scheme such that the induced map on k-points is the one discussed above, but that is a fact.

There is another way to provide $E(k)$ with a group structure. Fix again a point P_0 and prove that the map

$$E(k) \to \mathrm{Pic}^0(E) = \mathrm{Div}^0(E)/\mathrm{Prin}(E), \qquad P \mapsto [P] - [P_0],$$

is a bijective map. Thus, we can transport the group structure on $\mathrm{Pic}^0(E)$ to E. Show that this is the same group structure as defined above.

(41) The curve $x^3 + y^3 = z^3$ in $\mathbb{P}^2$ has genus 1 (assume that the characteristic of the field is not 3). Thus, it has a Weierstrass model. Following the proof given in class, working with the point $(1:0:1)$, produce this Weierstrass model.

(42) Let $\{P_1, \dots, P_6\}$ be 6 distinct points of $\mathbb{A}^1$. Consider the field extensions $k(x)[y_1]/(y_1^2 - (x-P_1)(x-P_2)(x-P_3)(x-P_4))$ and $k(x)[y_2]/(y_2^2 - (x-P_3)(x-P_4)(x-P_5)(x-P_6))$. Prove that each such field extension corresponds to an elliptic curve E_i and the inclusion of fields to maps of degree 2, $E_i \to \mathbb{P}^1$. Show that the fibre product $E_1 \times_{\mathbb{P}^1} E_2$ is a curve C as well; find its function field as an extension of $k(t)$ and prove that $k(C)$ is a Galois extension of $k(t)$ with Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. (The situation should remind you of the compositum of the extensions $\mathbb{Q}(\sqrt{10})$ and $\mathbb{Q}(\sqrt{15})$, say.) Conclude that there is a "hidden" third curve E_3 between C and $\mathbb{P}^1$. Find its function field and its genus. Find the ramification points and the degree for the morphisms $C \to E_i$.

(43) Let $f : C_1 \to C_2$ be a surjective morphism from a curve of genus 8 to a curve of genus 4. Prove that f is ramified at exactly two points of C_1 and that those points have distinct images in C_2.

(44) Continuing the previous question and assume that the characteristic of the field is not 2, to simplify. Given C_2 of genus 4 and two points on C_2, can you prove that there is a double cover of C_2 ramified precisely at the given points?? I suspect this question is too hard with the technology we currently have, but I'd be interested to see how far you can get with this question.

(45) We prove here Hurwitz's theorem. Let X be a smooth projective curve over an algebraically closed field k and let Γ be its group of automorphism. We are going to assume that Γ is finite; this is always true if $g(X) \geq 2$ and can be proven independently. We want to prove that if $g(X) \geq 2$ then

$$\sharp\Gamma \leq 84(g(X) - 1).$$

(a) Consider the fixed field of $k(X)$ under Γ; it corresponds to a smooth projective curve Y and $k(Y) = k(X)^\Gamma$. The inclusion $k(Y) \hookrightarrow k(X)$ therefore corresponds to a morphism $f : X \to Y$. Apply Hurwitz genus formula to conclude that

$$\frac{2g(X) - 2}{\deg(f)} = 2g(Y) - 2 + \frac{1}{\deg(f)} \sum_P (e_P - 1).$$

Use the fact that the cover $X \to Y$ is Galois to conclude that if $f(P) = f(P')$ then $e_P = e_{P'}$. Use that to simplify the formula above to a formula of the form

$$\frac{2g(X)-2}{\deg(f)} = 2g(Y) - 2 + \sum_i (1 - 1/r_i),$$

where the r_i are integers greater than 2.

(b) The problem now is number-theoretic. We have on the one hand $2g(y)-2+\sum_{i=1}^{s}(1-1/r_i) > 0$ and we seek the minimum of this quantity, where $s \geq 0$ and $r_i \geq 2$. Prove that the minimum value is $1/42$.

(c) Combine the estimates to deduce Hurwitz's theorem.

(46) Let A be a ring and S a graded ring such that $S^0 = A$. Prove that there is a morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(A)$. Take now $S = A[x_0, \ldots, x_n]$ and show that the fibre over $\mathfrak{p} \in \mathrm{Spec}(A)$ is $\mathrm{Proj}(\mathrm{Frac}(A/\mathfrak{p})[x_0, \ldots, x_n])$.

(47) Consider a curve E of genus 1 and its projective embedding into $\mathbb{P}^2$ we have constructed as an application of the Riemann-Roch theorem. Use the theory we have developed since to show that the morphism $E \to \mathbb{P}^2$ constructed using $\mathcal{O}_E(3[t])$ is a closed immersion.

(48) Prove the geometric characterization of generation by global section, separating points and separating tangents, stated in the language of linear systems in the notes (page 88, just before Example 7.5.3).

(49) Provide the missing details in Example 7.5.4. in the notes.