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EXERCISES FOR THE COURSE IN ALGEBRAIC GEOMETRY, FALL 2014

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(1) Define functors $G_i : \textbf{Set} \to \textbf{Top}$, by $G_1(X) = (X, 2^X)$ (every set is open) and $G_2(X) = (X, \mathscr{T}_{tr})$, where $\mathscr{T}_{tr}$ is the trivial topology consisting only of X and the empty set $\emptyset$. In both cases, $G_i(f) = f$ for a function $f : X \to Y$. Let $F : \textbf{Top} \to \textbf{Set}$ be the forgetful functor.

Prove that G_i are functors and that there are natural bijections

$$\mathrm{Hom}_{\mathrm{Set}}(F(X, \mathscr{T}), S) = \mathrm{Hom}_{\mathrm{Top}}(X, G_2(S)),$$

and

$$\mathrm{Hom}_{\mathrm{Set}}(S, F(X, \mathscr{T})) = \mathrm{Hom}_{\mathrm{Top}}(G_1(S), X).$$

(In first case we say (F, G_2) are an adjoint pair and in the second case we say that (G_1, F) are an adjoint pair.)

(2) An object A in a category $\mathbf{C}$ is called *initial* (resp. *final*) if for every object B in $\mathbf{C}$, $\mathrm{Hom}_{\mathrm{C}}(A, B)$ (resp. $\mathrm{Hom}_{\mathrm{C}}(B, A)$) is a singleton. Prove that an initial (resp. final) object, if it exists, is unique up to unique isomorphism. Determine if such objects exist in the following categories: **Set**, **Top**, $\mathbf{VS}_{\mathrm{k}}$, **Ring** (the categories of sets, topological spaces, vector spaces over a fixed field k and rings).

(3) Let $\mathbf{C}$ be a category with objects X_1, X_2, Z and morphisms $f : X_1 \to Z, g : X_2 \to Z$. Prove that if $X_1 \times_Z X_2$ exists it is unique up unique isomorphism (the product is taken relative to f, g). Show that $X_1 \times_Z X_2$ always exists in **Set**, **Top**, $\mathbf{VS}_{\mathrm{k}}$, **Ring**.

(4) Define $X_1 \times X_2$ using the diagram

$$\begin{tikzcd} X_1 \times X_2 \arrow[r, "g"] \arrow[d, "f"] & X_2 \\ X_1 & \end{tikzcd}$$

Show that if $\mathbf{C}$ has a final object Z then $X_1 \times X_2 = X_1 \times_Z X_2$, where equal means that there is a unique isomorphism between the two.

(5) Let $\mathbf{C}$ be a category in which finite products exist and there is a final object Z. First establish that if there are morphisms $f, g : Y \to G$ then there is a canonical morphism $(f, g) : Y \to G \times G$ such that composing with the projections $G \times G \to G$ we get back f and g. Similarly, if $f_i : A_i \to B_i$ are morphisms, there is a canonical morphism $f_1 \times f_2 : A_1 \times A_2 \to B_1 \times B_2$ (how would you charaterise it?). You'd want to establish some expected formulas, such as $(f_1 \times f_2) \circ (f, g) = (f_1 \circ f, f_2 \circ g), (f_1 \times f_2) \circ (h_1 \times h_2) = (f_1 \circ h_1) \times (f_2 \circ h_2)$. Further, you would need to show that there is a canonical isomorphism $G \times (G \times G) = (G \times G) \times G$.

A *group object* in $\mathbf{C}$ is an object G with given morphisms

$$m : G \times G \to G, \quad e : Z \to G, \quad \iota : G \to G,$$

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$$\begin{tikzcd} G \times G \times G \arrow[r, "1_G \times m"] \arrow[d, "m \times 1_G"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G, \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "1_G \times \iota"] \arrow[d] & G \times G \arrow[d, "m"] \\ Z \arrow[r, "e"] & G \end{tikzcd}$$

(and similarly with $\iota \times 1_G$),

$$\begin{tikzcd} Z \times G \arrow[r, "e \times 1_G"] \arrow[dr] & G \times G \arrow[d, "m"] \\ & G \end{tikzcd}$$

(and similarly with $1_G \times e$).

Given a group object G prove that its functor of points F_G is a functor $\mathbf{C} \to \mathbf{Group}$. Namely, for every Y an object of $\mathbf{C}$, $F_G(Y) = \mathrm{Hom}_{\mathbf{C}}(Y, G)$ is a group, and any morphism $Y_1 \to Y_2$ induces a group homomorphism $F_G(Y_2) \to F_G(Y_1)$.

Conversely, using Yoneda's lemma, prove that if G is an object of $\mathbf{C}$ such that F_G is a functor $\mathbf{C} \to \mathbf{Group}$ then G is a group object of $\mathbf{C}$.

Finally, show that a group object of **Set** is just a group and a group object of **Group** is an abelian group.

(6) Let R be a commutative ring and S a multiplicative set in R. Let $\ell : R \to R[S^{-1}]$ be the ring homomorphism from R to its localization at S. Give an example where ℓ is not injective. When is ℓ the zero map? When is ℓ surjective?

Consider the category whose objects are ring homomorphisms $f : R \to B$ such that $f(s)$ is an invertible element of B for every $s \in S$. A morphism h in this category is a commutative diagram

$$\begin{tikzcd} R \arrow[r, "f_1"] \arrow[dr, "f_2"] & B_1 \arrow[d, "h"] \\ & B_2. \end{tikzcd}$$

Prove that $\ell : R \to R[S^{-1}]$ is an initial object in this category.

(7) Prove that $I \mapsto I^e = I[S^{-1}]$ and $J \mapsto J^c = \ell^{-1}(J)$ provide a bijection

$$\{\text{prime ideals of } R \text{ disjoint from } S\} \quad \longleftrightarrow \left\{\text{prime ideals of } R[S^{-1}]\right\}.$$

(8) Let R be a commutative ring; L, M, N left R-modules. Prove that there are canonical isomorphisms:

- $R \otimes_R M \cong M$;
- $M \otimes_R N \cong N \otimes_R M$;
- $L \otimes_R (M \otimes_R N) \cong (L \otimes_R M) \otimes_R N$;
- $L \otimes_R (M \oplus N) \cong L \otimes_R M \oplus L \otimes_R N$.

(9) Prove the following isomorphism of tensor products (the isomorphisms are as rings)

- $R/I \otimes_R R/J \cong R/(I + J)$. So, for example, $\mathbb{Z}/m \cdot \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n \cdot \mathbb{Z} \cong \mathbb{Z}/\gcd(m, n) \cdot \mathbb{Z}$.
- $R[x_1, \dots, x_n] \otimes_R R[y_1, \dots, y_m] \cong R[x_1, \dots, x_n, y_1, \dots, y_m]$.

- $R[x_1, \dots, x_n]/I \otimes_R R[y_1, \dots, y_m]/J \cong R[x_1, \dots, x_n, y_1, \dots, y_m]/\langle I, J\rangle$. (On the right hand side we take the ideal generated by I and J).

(10) Let $\varphi : \mathscr{F}_1 \to \mathscr{F}_2$ be a homomorphism of sheaves of abelian groups on a topological space X. Prove that $U \mapsto \mathrm{Ker}(\varphi(U))$ is a sheaf. Prove that $U \mapsto \mathrm{Im}(\varphi(U))$ is a pre-sheaf. Give an example where it is not a sheaf.

(11) Let $f : X \to Y$ be a continuous map of topological spaces. Recall that we have defined f_* as a functor from the category of abelian sheaves on X to the category of abelian sheaves on Y and f^{-1} as a functor from the category of abelian sheaves on Y to the category of abelian sheaves on X. Show that there are natural morphisms $f^{-1}f_*\mathscr{F} \to \mathscr{F}$ and $\mathscr{G} \to f_*f^{-1}\mathscr{G}$. Use this to show that (f^{-1}, f_*) are an adjoint pair, namely,

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}).$$

(12) Prove that Mumford's classification of the ideals of Spec $\mathbb{Z}[x]$ (pp. 74-75) is correct.

(13) Prove Mumford's comment on page 68 regarding the quasi-compactness of $(\mathrm{Spec} R)_f$ and that if R is not noetherian it may happen that an open subset of Spec R is not quasi-compact.

(14) Prove that the following are equivalent
   (a) Spec R is disconnected.
   (b) There are elements e_1, e_2 in R such that $1 = e_1 + e_2$ and $e_1^2 = e_1, e_2^2 = e_2, e_1e_2 = 0$. Such elements are called orthogonal idempotents.
   (c) $R \cong R_1 \times R_2$, a product of two commutative rings.

(15) (This exercises is about Spec $\mathbb{Z}[x]$ and is taken from Mumford, page 75.) What is $V((p)) \cap V((f))$, f a $\mathbb{Q}$-irreducible polynomial? What is $V((f)) \cap V((g))$, f, g, distinct $\mathbb{Q}$-irreducible polynomials?

(16) (Exercise 2.2 from Hartshorne) Let $(X, \mathcal{O}_X)$ be a scheme and let $U \subseteq X$ be an open subset. Prove that $(U, \mathcal{O}_X|_U)$ is a scheme.

(17) (Exercise 2.3 from Hartshorne) *Reduced schemes.* A scheme $(X, \mathcal{O}_X)$ is called reduced if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.
   (a) Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every $x \in X$ the local ring $\mathcal{O}_{X,x}$ has no nilpotent elements.
   (b) Let $(X, \mathcal{O}_X)$ be a scheme and define $(\mathcal{O}_X)_{red}$ as the sheaf associated to the pre-sheaf $U \mapsto (\mathcal{O}_X(U))_{red}$, where for a commutative ring A we denote by A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathcal{O}_X)_{red})$ is a scheme, called the reduced scheme associated to X. Show that there is a natural morphism
   $$(X, (\mathcal{O}_X)_{red}) \to (X, \mathcal{O}_X),$$
   that is a homeomorphism on the level of topological spaces.
   (c) Let $f : X \to Y$ be a morphism of schemes and assume that X is reduced. Prove that f factors uniquely through $(Y, (\mathcal{O}_Y)_{red})$.

(18) (Exercise 2.18 from Hartshorne)
   (a) Let A be a ring and $X = \mathrm{Spec}(A)$. Let $f \in A$. Prove that X_f is empty if and only if f is nilpotent.
   (b) Let $\varphi : A \to B$ be a homomorphism of rings and $(f, f^*) : Y \to X$ the induced morphism of schemes, where $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B)$. Prove that φ is injective if and only if the map of sheaves f^* is injective (meaning, the sheaf $\mathrm{Ker}(f^*)$ is the zero sheaf). In that case, show further that f is dominant, meaning, $f(Y)$ is dense in X.
   (c) Show that if φ is surjective then f provides a homemorphism of Y onto a closed set of X and the map f^* is surjective, meaning, the sheaf $\mathrm{Im}(f^*)$ is equal to $f_*\mathcal{O}_Y$. (You may wish to reduce this statement to proving surjectivity on the level of stalks.)

(d) Prove the converse to (c). That is, for $f : Y \to X$ as above, such that $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, it follows that φ is surjective.

(19) Let k be a field. Prove that $U = \mathbb{A}^2_k - [(x, y)]$ - the plane minus the origin - is not an affine scheme.

(20) Let $A = \mathbb{A}^1_k$, where k is a field. Let $B = \mathrm{Spec}(k[x, y]/(y^2 - x^3)) - [(x, y)]$ - the cuspidal curve minus the cusp. Find the k-morphisms from A to B; show that there are "plenty" of morphisms from B to A. Is there an injective morphism from B to A? Is B isomorphic to an open subscheme of A? (Hint: consider the intersection of the line $y = \alpha x$ with the cuspidal curve and with the vertical line $x = -1$). Is there a surjective morphism from B to A?

(21) Let k be a field and S a scheme. Show that to give a morphism $\mathrm{Spec}(k) \to S$ is to give a point P of S and an inclusion $\mathbf{k}(P) \to k$.

(22) Let X_1, X_2 be two copies of the affine line over a field k. Let $U_i \subset X_i$ be the open subschemes that are the complements of the closed point $[(x)]$ (in the "naive" picture, we delete the origin of each affine line). Glue X_1 to X_2 by identifying U_1 with U_2 via the identity map. Prove that the union - "the affine line with the doubled origin" - is not an affine scheme.

(23) Give an example of a field k_0 with an algebraic closure k, and a scheme X_0 over $\mathrm{Spec}(k_0)$, such that the morphism $X = X_0 \times_{k_0} k \to X_0$ has infinite fibres.

(24) **Frobenius**. Let X be a scheme in characteristic p; namely, X is a scheme over $\mathbb{F}_p$, or, equivalently, $\mathcal{O}_X$ is a sheaf of $\mathbb{F}_p$ algebras. Define a morphism $F_{abs} : X \to X$ as being the identity on the level topological spaces and being the ring homomorphism $a \mapsto a^p$ for any open U and $a \in \Gamma(U, \mathcal{O}_X)$.

(a) Prove that indeed F_{abs} is a morphism of schemes.

(b) Show, by example, that F_{abs} need not be an isomorphism. (In fact, it very rarely is).

(c) Assume now that X is a scheme over a field k of characteristic p. To emphasize we write $s : X \to \mathrm{Spec}(k)$. Consider the cartesian diagram:

$$\begin{tikzcd}
X^{(p)} \arrow[r] \arrow[d] & X \arrow[d, "s"] \\
\mathrm{Spec}(k) \arrow[r, "F_{abs}"] & \mathrm{Spec}(k)
\end{tikzcd}$$

Note that F_{abs} on the bottom is nothing else then the morphism induced by the ring homomorphism $a \mapsto a^p$ of k. Here $X^{(p)}$ is, by definition, the fibre product over $\mathrm{Spec}(k)$ of X and $\mathrm{Spec}(k)$ relative to the indicated morphisms. Note that the morphism $X^{(p)} \to X$ is <u>not</u> a morphism of schemes over k. Now consider the following diagram:

$$\begin{tikzcd}
X \arrow[dr, dotted] \arrow[drr, "F_{abs}"] \arrow[ddr, "s"'] & & \\
& X^{(p)} \arrow[r] \arrow[d] & X \arrow[d, "s"] \\
& \mathrm{Spec}(k) \arrow[r, "F_{abs}"] & \mathrm{Spec}(k)
\end{tikzcd}$$

Prove that the outer solid arrows commute and so induce a unique morphism (the dotted arrow) $X \to X^{(p)}$ that we shall denote F_X. This is the Frobenius morphism of X. Prove that it is a morphism of schemes over k.

(d) Now consider the case where $X = \operatorname{Spec} k[x_1, \dots, x_n]/(\{f_j\})$. Given a polynomial $f = \sum a_I x^I \in k[x_1, \dots, x_n]$ let $f^\sigma = \sum a_I^p x^I$. Prove that $X^{(p)} = \operatorname{Spec} k[x_1, \dots, x_n]/(\{f_j^\sigma\})$ and determine all the morphisms appearing in the diagrams above explicitly. Finally, assuming that k is algebraically closed and thinking of the closed points of X as vectors $\{\alpha = (\alpha_1, \dots, \alpha_n) \in k^n : f_j(\alpha) = 0, \forall j\}$, prove that $F_X((\alpha_1, \dots, \alpha_n)) = (\alpha_1^p, \dots, \alpha_n^p)$.

(25) Let $\bar{\mathbb{Q}}$ denote an algebraic closure of $\mathbb{Q}$. What are the closed points of $\operatorname{Spec}(\mathbb{Q}[x, y])$ and how are they related to the closed points of $\operatorname{Spec}(\bar{\mathbb{Q}}[x, y])$?

(26) Let $f : X \to Y$ be a morphism of schemes and $Z \to Y$ a closed immersion. Prove that in the cartesian diagram below j is a closed immersion:

$$\begin{tikzcd} X_Z \arrow[r, "j"] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r] & Y. \end{tikzcd}$$

(27) We consider group schemes over the base $S = \operatorname{Spec}(R_0)$.

(a) Let H and G be a affine group schemes over S. Let $f : G \to H$ be a homomorphism of group schemes. Write the condition for that in terms of rings. Give then a formula for $\operatorname{Ker}(f)$.

(b) Show that $f : \mathbb{G}_m \to \mathbb{G}_m$, given on points by $r \mapsto r^N$, is a homomorphism of group schemes with kernel μ_N.

(c) Assume that $p = 0$ in R_0 is a prime number. Show that $f : \mathbb{G}_a \to \mathbb{G}_a$, given on points by $r \mapsto r^p$, is a homomorphism of group schemes and that in fact we have a commutative diagram

$$\begin{tikzcd} \mathbb{G}_a \arrow[r, "f"] \arrow[dr, "F"'] & \mathbb{G}_a \arrow[d, "="] \\ & \mathbb{G}_a^{(p)}, \end{tikzcd}$$

where F is the Frobenius morphism. Show that the kernel is α_p.

(28) Let k be a field of characteristic p. Construct a non-commutative group scheme G of order p^2 over $\operatorname{Spec}(k)$, with a closed immersion $G \to \mathrm{GL}_2/k$, using the following hint:

$$\begin{pmatrix} \mu_p & \alpha_p \\ 0 & 1 \end{pmatrix}.$$

(29) Let A be a commutative ring and M an A-module. Prove the following:

(a) If M is free then M is flat.

(b) Prove that if M is a flat A-module then for every A-algebra A', $M \otimes_A A'$ is a flat A'-module.

(c) Suppose that M is flat over A and A is a flat A_0-algebra. Prove that M is flat over A_0.

(d) Let $S \subset A$ be a multiplicative set. Prove that $A[S^{-1}]$ is flat over A.

(e) Prove that M is flat over A if and only if for all prime ideals $\mathfrak{p}$ of A, $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module.

(30) Let I be a directed set (that is, I is a partially ordered set such that for all $i, j \in I$ there is a $k \in I$ with $i \leq k, j \leq k$). Let $\{M_i, i \in I\}$ be flat A modules. Prove that $\varinjlim M_i$ is a flat A-module.

(31) Let A be a domain and M an A-module. Prove that if M is flat over A then M is torsion free. Prove that if A is a PID then every torsion free A-module is flat.

(32) Let M be an A-module with the following property (**P**):

(**P**) for all $m_1, \dots, m_n$ in M and $a_1, \dots, a_n$ in A such that $\sum_i a_i m_i = 0$, there are elements $m'_1, \dots, m'_k$ of M and $b_{ij} \in A$ such that
$$m_i = \textstyle\sum_{j=1}^k b_{ij} m'_j, \quad i = 1, \dots, n,$$
and such that for all j, $\sum_{i=1}^n a_i b_{ij} = 0$.

Prove that M is flat if and only if it has property (**P**). (Suggestion: think of the vector $(a_1, \dots, a_n)$ as defining a map of A modules $A^n \to A$.)

(33) Let k be a field. Consider the morphism $\mathbb{A}^2_k \to \mathbb{A}^2_k$ corresponding to the ring homomorphism $\varphi : k[x, y] \to k[x, y]$ determined by $\varphi(x) = x, \varphi(y) = xy$. Determine the maximal open set over which this morphism is flat.

(34) Let **C** be a category and F : **C** $\to$ **Sets** a representable contravariant functor. Say F is represented by X. Thus, we have an isomorphism $h_X \cong F$. The identity morphism $X \to X$ corresponds then to an element $t^{univ} \in F(X)$. Such an element is called a universal element. (It depends on the choice of isomorphism $h_X \cong F$). Prove that it has the following property. For every $t \in F(S)$, there is a **unique** morphism $f_t : S \to X$ such that $F(f_t) : F(X) \to F(S)$ takes t^{univ} to t. (Hint: don't look far - this is essentially a tautology.)

(a) Prove that the functor F : **Top** $\to$ **Sets** associating to a topological space its set of open subsets is representable and find a universal object. Prove that the restriction of this functor to the subcategory of Hausdorff spaces is not representable. (The idea of a universal object could be useful at this point. If it is representable, there should be a Hausdorff space X and an open subset t^{univ} of $F(X)$ such that...)

(b) Let A be a commutative ring with 1, $S = \mathrm{Spec}(A)$. Consider the functor F on schemes over S such that $F(T) = \Gamma(T, \mathcal{O}_T)$; namely, the functor that associates to an S-scheme its global regular functions. Check that this is indeed a functor and show that it is representable by $\mathbb{A}^1_A = \mathrm{Spec}(A[x])$.

(35) (The definition of a Hilbert scheme) Let S be a scheme. By a family of subschemes of $\mathbb{P}^n$ parameterized by S we understand a closed subscheme $Y \subseteq \mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times S$ such that the morphism $Y \to S$ is flat. Show that the
$$\mathfrak{H}ilb_{\mathbb{P}^n} : \textbf{Schemes} \to \textbf{Sets}, \qquad \mathfrak{H}ilb_{\mathbb{P}^n}(S) = \{Y \subseteq \mathbb{P}^n_S : Y \text{ flat over } S\},$$
associating to a scheme S the families of subschemes of $\mathbb{P}^n$ parameterized by S is indeed a functor.

One of Grothendieck's main theorems is that this functor, restricted to the category of locally noetherian schemes, is representable. That is, there is a locally noetherian scheme $\mathbb{H}\mathrm{ilb}_{\mathbb{P}^n}$ and a family $Y^{univ} \subseteq \mathbb{P}^n_{\mathbb{H}\mathrm{ilb}_{\mathbb{P}^n}} = \mathbb{P}^n_{\mathbb{Z}} \times \mathbb{H}\mathrm{ilb}_{\mathbb{P}^n}$ such that for every flat family $Y \subseteq \mathbb{P}^n_S$ there is a unique morphism $f : S \to \mathbb{H}\mathrm{ilb}_{\mathbb{P}^n}$ satisfying $f^*(Y^{univ}) = Y$.

Explain how this implies the following variant. Let k be a field. For a scheme S over k consider the functor parameterizing closed subschemes of $\mathbb{P}^n_S$ that are flat over S. Then this functor is representable by $\mathbb{H}\mathrm{ilb}_{\mathbb{P}^n} \times \mathrm{Spec}(k)$.

Now take $k = \mathbb{C}$ and show that the $\mathbb{C}$-scheme $\mathbb{H}\mathrm{ilb}_{\mathbb{P}^1} \times \mathrm{Spec}(\mathbb{C})$ is "huge" for example in the following sense. For every n construct a morphism $(\mathbb{A}^n_{\mathbb{C}} - Z) \to \mathbb{H}\mathrm{ilb}_{\mathbb{P}^1} \times \mathrm{Spec}(\mathbb{C})$ that is injective on closed points, where Z is the closed subscheme of $\mathbb{A}^n_{\mathbb{C}}$ where 2 coordinates are equal.

(36) Prove or disprove: Let $X \to S$ be a morphism of schemes then $X \times_S X \to X$ is a flat morphism.

(37) Let k be a field. Prove that the line over k with the double origin is not a separated scheme by showing that the valuative criterion for separatedness fails.

(38) Let k be a field. Prove that the morphism $\mathbb{A}^1_k \to \mathrm{Spec}(k)$ is not proper by showing that the valuative criterion for properness fails. Prove, as well, that the morphism $\mathbb{A}^2_k \to \mathbb{A}^1_k$ is not proper.

(39) Let A be a ring. Prove that the morphism $\mathbb{P}^1_A \to \mathrm{Spec}(A)$ is proper using the valuative criterion for properness.