ASSIGNMENT 6 - MATH576, 2006

Solve the following questions. Submit by Monday, November 13.

(1) (a) Let $X$ be a topological space and $G$ a finite group of automorphisms of $X$ operating with no fixed points. Namely, if $g(x) = x$ for some $x \in X$ then $g = 1_G$. Let $X/G$ be the quotient space for the equivalence relation whose equivalence classes are the orbits of $G$ in $X$. Prove that if $X$ is $T_2$ the map

$$X \to X/G$$

is a covering map. Prove that if $X$ is connected and locally pathwise connected then so is $X/G$.

(b) If $X$ is simply connected prove that $\pi_1(X/G) \cong G$ (as groups, of course).

(c) We want to apply the preceding to the topological space $S^3$. We think about $S^3$ as $\{ (z_1, z_2) : z_i \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1 \}$. Let $n, k$ be two relatively prime positive integers. Apply the above to the group of automorphisms $\langle h \rangle$ generated by

$$h(z_1, z_2) = (z_1 e^{2\pi i/n}, z_2 e^{2\pi ik/n}),$$

to deduce that the orbit space $S^3/\langle h \rangle$ (called the lens space $L(n, k)$) has fundamental group $\mathbb{Z}/n\mathbb{Z}$.

(2) This exercise deals with a “bouquet of two circles”. This is the quotient space $X$ of two copies of $S^1$ obtained by identifying the two points, one on each copy, $(1, 0)$. (See figure.)

(a) Find three different covering spaces $p : E \to X$ of degree 2. (The answer is just a drawing with an explanation of the map.) For each find a closed loop $\gamma$ such that $\gamma$ does not lift to a closed loop but $\gamma^2$ does.

(b) Find two different infinite covers $p : E \to X$. (Again, pictures will do.)

(c) Find the universal covering space of $X$. (Again, a picture will do, but explain carefully the map!)

(d) Assuming that the fundamental group of $X$, say with respect to the singular point $x$, is the free group on two generators $\alpha, \beta$ ($\alpha$ corresponds to one of the
circle, $\beta$ to the other) find the subgroups corresponding to each one of the coverings you found above.

(e) **Bonus.** Prove the following useful fact: *to give a subgroup of index $n$ of a group $G$ is to give a transitive action of $G$ on a set $S$ of $n$ elements and an element $s \in S$.*

Use this, or any other method, to find a non-normal subgroup $H$ of index 3 of $\pi(X, x)$. Find a covering space $p : X_H \to X$ such $p_*\pi(X_H, *) = H$.

(3) (a) Let $p : E \to B$ be a covering space. Prove that for $n > 1$ we have

$$\pi_n(E, e_0) \cong \pi_n(B, b_0).$$

(b) Show that the higher homotopy groups of $S^1$ are trivial.