ASSIGNMENT 6 - MATH576, 2006

Solve the following questions. Submit by Monday, November 13.

(1) (a) Let X be a topological space and G a finite group of automorphisms of X operating with no fixed points. Namely, if $g(x) = x$ for some $x \in X$ then $g = 1_G$. Let X/G be the quotient space for the equivalence relation whose equivalence classes are the orbits of G in X. Prove that if X is T_2 the map

$$X \to X/G$$

is a covering map. Prove that if X is connected and locally pathwise connected then so is X/G.

(b) If X is simply connected prove that $\pi_1(X/G) \cong G$ (as groups, of course).

(c) We want to apply the preceding to the topological space S^3. We think about S^3 as $\{(z_1, z_2) : z_i \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1\}$. Let n, k be two relatively prime positive integers. Apply the above to the group of automorphisms $\langle h \rangle$ generated by

$$h(z_1, z_2) = (z_1 e^{2\pi i/n}, z_2 e^{2\pi i k/n}),$$

to deduce that the orbit space $S^3/\langle h \rangle$ (called the *lens space* $L(n, k)$) has fundamental group $\mathbb{Z}/n\mathbb{Z}$.

(2) This exercise deals with a "bouquet of two circles". This is the quotient space X of two copies of S^1 obtained by identifying the two points, one on each copy, $(1, 0)$. (See figure.)

(a) Find three different covering spaces $p : E \to X$ of degree 2. (The answer is just a drawing with an explanation of the map.) For each find a closed loop γ such that γ does not lift to a closed loop but γ^2 does.

(b) Find two different infinite covers $p : E \to X$. (Again, pictures will do.)

(c) Find the universal covering space of X. (Again, a picture will do, but explain carefully the map!)

(d) Assuming that the fundamental group of X, say with respect to the singular point x, is the free group on two generators α, β (α corresponds to one of the circle, β to the other) find the subgroups corresponding to each one of the coverings you found above.

(e) **Bonus.** Prove the following useful fact: *to give a subgroup of index n of a group G is to give a transitive action of G on a set S of n elements and an element $s \in S$.*
Use this, or any other method, to find a non-normal subgroup H of index 3 of $\pi(X, x)$. Find a covering space $p : X_H \to X$ such $p_*\pi(X_H, *) = H$.

(3) (a) Let $p : E \to B$ be a covering space. Prove that for $n > 1$ we have

$$\pi_n(E, e_0) \cong \pi_n(B, b_0).$$

(b) Show that the higher homotopy groups of S^1 are trivial.