(1) Let $S^n := \{ x \in \mathbb{R}^{n+1} : \|x\| = 1 \}$ be the $n$-dimensional sphere. It is the boundary of the $n+1$ dimensional ball $B^{n+1} := \{ x \in \mathbb{R}^{n+1} : \|x\| \leq 1 \}$.

Prove that $S^n$ is homeomorphic to two copies of $B^n$ glued along their boundary. To be precise, let $X$ be $\{ \alpha \} \times B^n \cup \{ \beta \} \times B^n$ with the obvious topology (the one inducing on each copy of $B^n$ its natural topology). Define an equivalence relation on $X$ by identifying $(\alpha, x)$ with $(\beta, x)$ when $\|x\| = 1$. (The other equivalence classes are one point sets). Denote this relation by $\sim$. Prove that $S^n \cong X/\sim$.

(2) Do Exercise 2, page 144.

(3) Do Exercise 13, page 172.

(4) Let $F = \mathbb{R}$ or $\mathbb{C}$. Define an equivalence relation on $F^{n+1} - \{0\}$ by identifying $x$ with $\lambda x$ for any $\lambda \in F^\times$. The quotient space is called the $n$-dimensional (real or complex) projective space and we shall denote it by $P^n(F)$ (though in topology we often find the notation $FP^n$). Prove that $P^n(F)$ is Hausdorff and compact. (It is also isomorphic to $GL_n(F)/P(F)$ where $P$ is the parabolic subgroup of $GL_n$ consisting of matrices $A = (a_{ij})$, with $a_{21} = a_{31} = \cdots = a_{n1} = 0$. You don’t have to prove it if you don’t use it.)

Prove also that $P^1(\mathbb{R}) \cong S^1$ and $P^1(\mathbb{C}) \cong S^2$.

Remark: There are probably many ways to solve this exercise. I think that the hardest point is compactness. You may want to argue by induction, showing first that one has a decomposition:

$$P^n(F) = F^n \coprod P^{n-1}(F),$$

obtained by, say, distinguishing whether the first coordinate is zero or not. Then you can prove that a collection of open sets covering the $P^{n-1}(F)$ part will “overflow a lot” into the $F^n$ part.