October 9, 2006

ASSIGNMENT 4 - MATH576, 2006

Solve the following questions. Submit by Monday, October 16.

(1) Let $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ be the n-dimensional sphere. It is the boundary of the $n+1$ dimensional ball $\mathbb{B}^{n+1} := \{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\}$.

Prove that $\mathbb{S}^n$ is homeomorphic to two copies of $\mathbb{B}^n$ glued along their boundary. To be precise, let X be $\{\alpha\} \times \mathbb{B}^n \cup \{\beta\} \times \mathbb{B}^n$ with the obvious topology (the one inducing on each copy of $\mathbb{B}^n$ its natural topology). Define an equivalence relation on X by identifying (α, x) with (β, x) when $\|x\| = 1$. (The other equivalence classes are one point sets). Denote this relation by $\sim$. Prove that

$$\mathbb{S}^n \cong X/\sim .$$

(2) Do Exercise 2, page 144.

(3) Do Exercise 13, page 172.

(4) Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Define an equivalence relation on $\mathbb{F}^{n+1} - \{0\}$ by identifying x with λx for any $\lambda \in \mathbb{F}^\times$. The quotient space is called the n-dimensional (real or complex) projective space and we shall denote it by $\mathbb{P}^n(\mathbb{F})$ (though in topology we often find the notation $\mathbb{F}\mathbb{P}^n$). Prove that $\mathbb{P}^n(\mathbb{F})$ is Hausdorff and compact. (It is also isomorphic to $\mathrm{GL}_n(\mathbb{F})/P(\mathbb{F})$ where P is the parabolic subgroup of GL_n consisting of matrices $A = (a_{ij})$, with $a_{21} = a_{31} = \cdots = a_{n1} = 0$. You don't have to prove it if you don't use it.)

Prove also that $\mathbb{P}^1(\mathbb{R}) \cong S^1$ and $\mathbb{P}^1(\mathbb{C}) \cong \mathbb{S}^2$.

Remark: There are probably many ways to solve this exercise. I think that the hardest point is compactness. You may want to argue by induction, showing first that one has a decomposition:

$$\mathbb{P}^n(\mathbb{F}) = \mathbb{F}^n \coprod \mathbb{P}^{n-1}(\mathbb{F}),$$

obtained by, say, distinguishing whether the first coordinate is zero or not. Then you can prove that a collection of open sets covering the $\mathbb{P}^{n-1}(\mathbb{F})$ part will "overflow a lot" into the $\mathbb{F}^n$ part.