September 10, 2006

ASSIGNMENT 2 - MATH576, 2006

Solve the following questions. Submit by Monday, September 25.

(1) Solve questions 11 and 13 on page 101.[1]

(2) Solve questions 6, 7, 13 on page 111.

(3) Prove that $(0,1) \not\cong [0,1]$. (Both with the subspace topology induced from $\mathbb{R}$.)

(4) Prove that $\{(x,y) : x^2 + y^2 < 1\} \cong \mathbb{R}^2$, where the open circle has the subspace topology.

(5) Read about the box topology in §19 and answer question 7, page 118.

(6) Find a metric space and two open balls in it such that the ball with the smaller radius contains the ball with the bigger one and does not coincide with it.

(7) A metric is called an ultrametric, or a non-archimedean metric, if it satisfies the inequality $d(x,y) \le \max(d(x,z), d(z,y))$. Prove that in an ultrametric topological space the following holds:

  (a) Every point in a ball is its center.

  (b) Any triangle is isoceles.

  (c) Spheres are both open and close.

  (d) Prove that the following is an ultrametric on $\mathbb{Q}$: Fix a prime number p. For a rational number $q = p^r \frac{a}{b}$, with $r \in \mathbb{Z}, (a,b) = 1, a, b \in \mathbb{Z}$ and $p \nmid ab$, define $d(q,0) = p^{-r}$. More generally, for two rational numbers q_1, q_2 define

$$d(q_1, q_2) = d(q_1 - q_2, 0).$$

  We call this metric the p-adic metric on $\mathbb{Q}$.

  (e) Show that the sequence $p, p^2, p^3, p^4, \ldots$ converges in $\mathbb{Q}$ under the p-adic topology and find its limit.

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[1] We always refer to the second edition of the textbook!