ASSIGNMENT 2 – ELEMENTARY NUMBER THEORY MATH 392

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Answer the following questions:

1. By a suitable variation on Euclid's proof, prove that there are infinitely many primes p, such that $p \equiv 3 \pmod 4$.

2. Prove that e, defined as

$$e = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots,$$

is an irrational number.

3. Using the methods of the lecture, define the g.c.d. of n natural numbers $a_1, \cdots, a_n$ as the unique natural number d such that $d|a_i$ for every i and if $e|a_i$ for every i then $e|d$. Lets denote is by

$$(a_1, \cdots, a_n).$$

Prove that it exists and unique. Show also that there exists integers $x_1, \cdots, x_n$ such that

$$x_1 a_1 + \cdots + x_n a_n = (a_1, \cdots, a_n).$$

Prove the identity

$$(a_1, \cdots, a_n) = ((a_1, \cdots, a_{n-1}), a_n).$$

Now do exercise (ii) on page 7 in Baker's book:
" Find integers x, y, z such that $35x + 55y + 77z = 1$."

4. (a) Prove that for every prime p and every $1 \leq r \leq p-1$, $p| \binom{p}{r}$.

(b) Let f be a numerical polynomial of degree n. Let p be a prime greater then n. Prove that $p|(f(p) - f(0))$.

5. Compute a formula for

$$1^4 + 2^4 + \cdots + (x-1)^4.$$

6. Evaluate $\sum_{d|n} \mu(n/d)\sigma(d)$.

7. Do exercise (i) on page 16 of Baker's book:
" Evaluate $\sum_{d|n} \mu(d)\sigma(d)$ in terms of the distinct prime divisors of n."

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\phi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8
$\mu(n)$	1	-1	-1	0	-1	1	-1	0	0	1	-1	0	-1	1	1
$d(n)$	1	2	2	3	2	4	2	4	3	4	2	6	2	4	4
$\sigma(n)$	1	3	4	7	6	12	8	15	13	18	12	28	14	24	24

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\phi(n)$	8	16	6	18	8	12	10	22	8	20	12	18	12	28	8
$\mu(n)$	0	-1	0	-1	0	1	1	-1	0	0	1	0	0	-1	-1
$d(n)$	5	2	6	2	6	4	4	2	8	3	4	4	6	2	8
$\sigma(n)$	31	18	39	20	42	32	36	24	60	31	42	40	56	30	72

n	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
$\phi(n)$	30	16	20	16	24	12	36	18	24	16	40	12	42	20	24
$\mu(n)$	-1	0	1	1	1	0	-1	1	1	0	-1	-1	-1	0	0
$d(n)$	2	6	4	4	4	9	2	4	4	8	2	8	2	6	6
$\sigma(n)$	32	63	48	54	48	91	38	60	56	90	42	96	44	84	78

n	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
$\phi(n)$	22	46	16	42	20	32	24	52	18	40	24	36	28	58	16
$\mu(n)$	1	-1	0	0	0	1	0	-1	0	1	0	1	1	-1	0
$d(n)$	4	2	10	3	6	4	6	2	8	4	8	4	4	2	12
$\sigma(n)$	72	48	124	57	93	72	98	54	120	72	120	80	90	60	168