ASSIGNMENT 8 - MATH 251, WINTER 2007

Submit by Monday, March 19, 12:00

In this field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

1. Consider results of an experiment given by a series of points:

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),$$

where $x_1 < x_2 < \cdots < x_n$ and the x_i, y_i are real numbers.

We assume that the actual law governing this data is linear. Namely, that there is an equation of the form $f_{A,B}(x) = Ax + B$ that fits the data up to experimental errors. Therefore, we look for such an equation $Ax + B$ that fits the data best. Our measure for that is "the method of list squares". Namely, given a line $Ax + B$, let $d_i = |y_i - (Ax_i + B)|$ (the distance between the theoretical y and the observed y). Then we seek to minimize

$$d_1^2 + d_2^2 + \cdots + d_n^2.$$

Let

$$T : \mathbb{R}^2 \to \mathbb{R}^n,$$

be the map

$$T(A, B) = (f_{A,B}(x_1), \ldots, f_{A,B}(x_n)).$$

Prove that T is a linear map and that the problem we seek to solve is to minimize

$$\|T(A, B) - (y_1, \ldots, y_n)\|^2.$$

Let W be the subspace of $\mathbb{R}^n$ which is the image of T. Prove that W is two dimensional and that $\{s_1, s_2\}$ is a basis for W, where $s_1 = (1, 1, \ldots, 1), s_2 = (x_1, x_2, \ldots, x_n)$.

Assume for simplicity that $\sum_{i=1}^n x_i = 0$ (this can always be achieved by shifting the data). Find an orthonormal basis for W and use it to find the vector in W closest to $(y_1, \ldots, y_n)$.

Put now everything together to get explicit formulas for A, B such that $f_{A,B}(x)$ is the best linear approximation to the data

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n).$$

(Still under the assumption $\sum_{i=1}^n x_i = 0$.)

2. Angles. Let V be an inner product space over $\mathbb{R}$. Define the angle θ between two non-zero vectors u, v in V to be the unique angle $\theta, 0 \le \theta \le \pi$ such that

$$\cos(\theta) = \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|}.$$

This is well-defined by Cauchy-Schwartz. Prove the Law of cosines holds *with this definition* of $\cos(\theta)$: That is, consider a triangle with sides A, B, C of lengths a, b, c, respectively, and let θ be the angle between A and B in the sense defined above. Then

$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

Deduce from the fact that the law of cosines holds in plane geometry for angles defined in the usual way, that our definition of an angle *generalizes* the usual definition.

3. Let W be the subspace of $\mathbb{F}^4$ defined by the equation $x_1 + x_2 + x_3 + x_4 = 0$. Find the orthogonal projection of $(1, 0, 0, 0)$ on W.

4. Given a monic polynomial of degree n, $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$, show that there exists an $n \times n$ matrix A, such that

$$\Delta_A(t) = f(t).$$

(Suggestion: Look at past assignments.)

5. Find an explicit formula for a_n in the series $a_0, a_1, a_2, \dots$ given by

$$0,\ 1,\ 1,\ 3,\ 5,\ 11,\ 21,\ 43,\ \dots.$$

6. **Markov Processes.** Imagine a particle that can be in any one of n states

$$S_1, S_2, \dots, S_n.$$

(These may be the spin states of a particle, or the room my kid is in...) Initially, we might not know exactly the state the particle is in and have at our disposal only the probability it is in a certain state, that is, we have a *probability distribution*

$$(f_1, f_2, \dots, f_n), \quad f_i \geq 0, \quad f_1 + f_2 + \cdots + f_n = 1.$$

A certain process is now taking place (it is an example of certain stochastic processes called *Markov chains*). At every time $t = 1, 2, 3, \dots$ the particle may change its state. We know the probability it change its state from state S_j to state S_i and construct a matrix $M = (m_{ij})$, where m_{ij} is the probability that the particle goes from state S_j to state S_i.

(1) Show that M is a matrix of non-negative real numbers such that for every j we have $\sum_{i=1}^n m_{ij} = 1$.

(2) Prove that the probability distribution describing the state of the particle at time n is given by

$$M^n \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

(3) Consider the case of two states S_1, S_2 where M (called the *transition matrix*) is given by

$$\begin{pmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{pmatrix}.$$

Show that for any initial probability distribution (f_1, f_2) the limit

$$\lim_{n\to\infty} M^n \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

exists, and compute it.

7. Find the characteristic polynomial, eigenvalues λ and bases for the spaces E_λ for the following matrices:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix},$$

over the fields: (i) $\mathbb{R}$, (ii) $\mathbb{C}$, (iii) $\mathbb{Z}/2\mathbb{Z}$, (iv) $\mathbb{Z}/5\mathbb{Z}$.