ASSIGNMENT 7 - MATH 251, WINTER 2007

Submit by Monday, March 12, 12:00

In this assignment \(V\) is an inner product space over \(\mathbb{F}\), where \(\mathbb{F} = \mathbb{R}\) or \(\mathbb{C}\).

1. Prove the Parallelogram Law:
\[
\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).
\]

2. Prove that any inner product on \(\mathbb{F}^n\) arises from a positive definite Hermitian matrix.

3. Let \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) be a matrix of complex numbers. Prove that the function
\[
\langle (x_1, x_2), (y_1, y_2) \rangle = (x_1, x_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},
\]
is an inner product on \(\mathbb{C}^2\) if and only if \(a\) and \(d\) are positive real numbers, \(c = \overline{b}\) and \(ad - bc > 0\).

In the case of the matrix \(\begin{pmatrix} 1 & 1+i \\ 1-i & 5 \end{pmatrix}\) compute \(\langle (1, 2), (3, 4) \rangle\) and \(\| (2, 5i) \|\).

4. Let \(a < b\) be real numbers. Show that the function
\[
\langle f, g \rangle = \int_a^b f(x)g(x)\,dx
\]
defines an inner product of \(\mathbb{R}[x]_n\). Compute the norm of the vector \(f(x) = 1 + x + x^2\) in the case \((a, b) = (0, 1)\) and in the case \((a, b) = (0, 2)\).

5. Find an orthonormal basis for \(\mathbb{C}^2\) with the inner product defined by the matrix \(\begin{pmatrix} 1 & 1+i \\ 1-i & 5 \end{pmatrix}\).

\(^1\) This is also true for \(\mathbb{C}[x]_n\) if we define
\[
\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\,dx.
\]

Note that in this case one can do the integration formally because we are dealing with polynomials. Thus, for example,
\[
\langle x^2, 1 + ix \rangle = \int_a^b (x^2 - ix^3) \, dx = (x^3/3 - ix^4/4)\big|_a^b = (b^3/3 - ib^4/4) - (a^3/3 - ia^4/4).
\]
6. Perform the Gram-Schmidt process for the basis \( \{1, x, x^2\} \) to \( \mathbb{R}[x]_2 \) with respect to the inner product
\[
\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx.
\]

7. Let \( \langle \cdot, \cdot \rangle \) be an inner product on \( \mathbb{R}^n \). Define the unit ball in \( \mathbb{R}^n \) to be the set
\[
\mathbb{B} = \{(x_1, \ldots, x_n) : \|(x_1, \ldots, x_n)\| \leq 1\}.
\]
Show that this is a central convex body:
1. \( 0 \in \mathbb{B} \).
2. if \( \alpha \in [0, 1] \) and \( u, v \in \mathbb{B} \) then \( \alpha u + (1 - \alpha)v \in \mathbb{B} \);
3. if \( |\alpha| \leq 1 \) then \( v \in \mathbb{B} \Rightarrow \alpha v \in \mathbb{B} \).

8. Prove Pythagoras’s theorem: Let \( V \) be an inner product space and \( u, v \) two orthogonal vectors in \( V \) then
\[
\|u + v\|^2 = \|u\|^2 + \|v\|^2.
\]