ASSIGNMENT 6 - MATH 251, WINTER 2007

Submit by Monday, March 5, 12:00

1. Let $T : V \to W$ be a linear map and define $T^* : W^* \to V^*$ by $(T^*(g))(v) := g(Tv)$. Prove the following lemma:

LEMMA **1.** (1) *T^* is a well-defined linear map.*

(2) *Let B, C be bases to V, W, respectively. Let $A = {}_C[T]_B$ be the $m \times n$ matrix representing T, where $n = \dim(V), m = \dim(W)$. Then the matrix representing T^* with respect to the dual bases B^*, C^* is the transpose of A:*

$$ {}_{B^*}[T^*]_{C^*} = {}_C[T]_B{}^t. $$

(3) *If T is injective then T^* is surjective. (Do NOT use Proposition 7.2.8 in the notes).*

(4) *If T is surjective then T^* is injective.*

2. Let $V = \mathbb{R}[t]_n$ be the space of polynomials with real coefficients of degree at most n. Let $r_0 < r_1 < \cdots < r_n$ be $n+1$ distinct real numbers. We have seen that

$$ f_i : \mathbb{R}[t]_n \to \mathbb{R}, \qquad f_i(g(t)) = g(r_i), $$

is a linear map and so is a linear functional. Prove that

$$ B^* = \{f_0, \dots, f_n\} $$

is a basis for V^*. Find the basis $B = \{g_0, g_1, \dots, g_n\}$ of V dual to it. Prove that if $g \in V$ then

$$ g(t) = \sum_{i=0}^{n} g(r_i) \cdot g_i(t). $$

3. Consider a system of linear equations over a field $\mathbb{F}$:

$$ \begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{array} \tag{0.1} $$

View this as applying the functionals $(a_{11}, \dots, a_{1n}),\ (a_{21}, \dots, a_{2n}), \dots (a_{m1}, \dots, a_{mn})$ to the vector $(x_1, \dots, x_n)$. Under this interpretation show that the solutions to the homogenous system are $U^\perp$, where U is the row space of A. Conclude that the dimension of solutions to the homogenous system is $n - \mathrm{rank}_r(A)$. (Remark: we know that already, but this is just one more way to think about it.)

4. **Latin Squares.** Wikipedia has an entry for Latin squares which you may find interesting.

A Latin square is, for us, an $n \times n$ matrix all whose entries are integers belonging to $\{1, 2, \dots, n\}$ in such a way that every row and every column contain every number exactly once. For example,

1	2
2	1

1	2	3
3	1	2
2	3	1

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

Such matrices are important for group theory, experimental designs and linear algebra; there are many open questions. It is a hard theorem that there are more than $(n!)^{2n}/n^{n^2}$ Latin squares of order n (this is more than exponential in n). One way to construct Latin squares is as multiplication tables for groups.

If G is a group of order n we write its elements $a_1, \ldots, a_n$ and the ij entry of the table is k if $a_i a_j = a_k$. I'll let you ponder, if you are interested, why this gives a Latin square.

Here is an interesting way to construct Latin squares that has to do with linear algebra. Let $\mathbb{F}$ be a field with q elements and choose in $\mathbb{P}^2(\mathbb{F})$ three distinct points x, y, z lying on a line ℓ. Enumerate the lines through x, besides the line ℓ, by the numbers $1, \ldots, q$ (can we do that?). Do the same for y and z. Define a matrix $M = (m_{ij})$ as follows. Let t be the intersection point of the i-th line through x and the j-th line through y. The line connecting t to z is different from ℓ (why?) and so has a certain number k. Let $m_{ij} = k$. Prove that this works. Namely that this is a well-defined process yielding a Latin square for $\{1, 2, \ldots, q\}$.