1. Let $T : V \to W$ be a linear map and define $T^* : W^* \to V^*$ by $(T^*(g))(v) := g(Tv)$. Prove the following lemma:

**Lemma 1.**

1. $T^*$ is a well-defined linear map.
2. Let $B, C$ be bases to $V, W$, respectively. Let $A = c[T]_B^C$ be the $m \times n$ matrix representing $T$, where $n = \dim(V), m = \dim(W)$. Then the matrix representing $T^*$ with respect to the dual bases $B^*, C^*$ is the transpose of $A$:

   \[ B^*[T^*]_{C^*} = c[T]^t. \]

3. If $T$ is injective then $T^*$ is surjective. (Do NOT use Proposition 7.2.8 in the notes).
4. If $T$ is surjective then $T^*$ is injective.

2. Let $V = \mathbb{R}[t]^n$ be the space of polynomials with real coefficients of degree at most $n$. Let $r_0 < r_1 < \cdots < r_n$ be $n + 1$ distinct real numbers. We have seen that $f_i : \mathbb{R}[t]^n \to \mathbb{R}, f_i(g(t)) = g(r_i)$, is a linear map and so is a linear functional. Prove that $B^* = \{f_0, \ldots, f_n\}$ is a basis for $V^*$. Find the basis $B = \{g_0, g_1, \ldots, g_n\}$ of $V$ dual to it. Prove that if $g \in V$ then

   \[ g(t) = \sum_{i=0}^{n} g(r_i) \cdot g_i(t). \]

3. Consider a system of linear equations over a field $F$:

   \begin{align*}
   a_{11}x_1 + \cdots + a_{1n}x_n \\
   \vdots \\
   a_{m1}x_1 + \cdots + a_{mn}x_n
   \end{align*}

   (0.1)

   View this as applying the functionals $(a_{11}, \ldots, a_{1n}), (a_{21}, \ldots, a_{2n}), \ldots (a_{m1}, \ldots, a_{mn})$ to the vector $(x_1, \ldots, x_n)$. Under this interpretation show that the solutions to the homogenous system are $U^\perp$, where $U$ is the row space of $A$. Conclude that the dimension of solutions to the homogenous system is $n - \text{rank}_r(A)$. (Remark: we know that already, but this is just one more way to think about it.)

4. **Latin Squares.** Wikipedia has an entry for Latin squares which you may find interesting.

   A Latin square is, for us, an $n \times n$ matrix all whose entries are integers belonging to $\{1, 2, \ldots, n\}$ in such a way that every row and every column contain every number exactly once. For example,

   \[
   \begin{array}{cccc}
   1 & 2 & 3 & 4 \\
   2 & 1 & 3 & 4 \\
   3 & 4 & 1 & 2 \\
   4 & 3 & 2 & 1
   \end{array}
   \]

   Such matrices are important for group theory, experimental designs and linear algebra; there are many open questions. It is a hard theorem that there are more than $(n!)^{2n}/n^{n^2}$ Latin squares of order $n$ (this is more than exponential in $n$). One way to construct Latin squares is as multiplication tables for groups.
If $G$ is a group of order $n$ we write its elements $a_1, \ldots, a_n$ and the $ij$ entry of the table is $k$ if $a_i a_j = a_k$. I’ll let you ponder, if you are interested, why this gives a Latin square.

Here is an interesting way to construct Latin squares that has to do with linear algebra. Let $\mathbb{F}$ be a field with $q$ elements and choose in $\mathbb{P}^2(\mathbb{F})$ three distinct points $x, y, z$ lying on a line $\ell$. Enumerate the lines through $x$, besides the line $\ell$, by the numbers $1, \ldots, q$ (can we do that?). Do the same for $y$ and $z$. Define a matrix $M = (m_{ij})$ as follows. Let $t$ be the intersection point of the $i$-th line through $x$ and the $j$-th line through $y$. The line connecting $t$ to $z$ is different from $\ell$ (why?) and so has a certain number $k$. Let $m_{ij} = k$. Prove that this works. Namely that this is a well-defined process yielding a Latin square for \{1, 2, \ldots, q\}.