ASSIGNMENT 4 - MATH 251, WINTER 2007

Submit by Monday, February 12, 12:00

1. Deduce from the theorems on determinants the following:
   (1) If a column is zero, the determinant is zero.
   (2) $\det(A) = \det(A^t)$, where A^t is the transposed matrix.
   (3) If a row is zero, the determinant is zero.
   (4) Let A be a matrix in "upper diagonal block form":
$$A = \begin{pmatrix} A_1 & & & \star \\ 0 & A_2 & & \\ & & \ddots & \\ 0 & & 0 & A_k \end{pmatrix}.$$
   Here each A_i is a square matrix say of size r_i, and A_2 starts at the r_1+1 column and r_1+1 row, etc. Prove that
$$\det(A) = \det(A_1)\det(A_2)\cdots\det(A_k).$$
   Conclude that the determinant of an upper triangular matrix is given by
$$\det \begin{pmatrix} a_{11} & & & \star \\ 0 & a_{22} & & \\ & & \ddots & \\ 0 & & 0 & a_{kk} \end{pmatrix} = a_{11}a_{22}\cdots a_{kk}.$$
   (Here each a_{ii} is a scalar).
2. Calculate the following series of determinants.
   (1) $\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\det\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $\cdots$
   (2) $\det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\det\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\det\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $\det\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$, $\cdots$
   (3) $\det\begin{pmatrix} x & -a_2 \\ 1 & x+a_1 \end{pmatrix}$, $\det\begin{pmatrix} x & 0 & a_3 \\ 1 & x & -a_2 \\ 0 & 1 & x+a_1 \end{pmatrix}$, $\det\begin{pmatrix} x & 0 & 0 & -a_4 \\ 1 & x & 0 & a_3 \\ 0 & 1 & x & -a_2 \\ 0 & 0 & 1 & x+a_1 \end{pmatrix}$, $\cdots$
3. Prove the following formula (the *Vandermonde determinant*):
$$\det\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots x_1^{n-1} \\ \vdots & & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots x_n^{n-1} \end{pmatrix} = \prod_{i>j}(x_i - x_j)$$

For example, for $n = 2, 3$ we have
$$\det\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} = (x_2 - x_1), \quad \det\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

4. *Reed-Solomon Codes.* Let $\mathbb{F}$ be a finite field with q elements. List the non-zero elements of $\mathbb{F}_q$ as $\{\beta_1, \dots, \beta_{q-1}\}$. Define a map

$$\mathbb{F}_q[x]_{k-1} \to \mathbb{F}^{q-1},$$

by

$$f \mapsto T(f) := (f(\beta_1), \dots, f(\beta_{q-1})).$$

Prove that T is a linear map and find when is it injective. When this holds, the image of T is a k-dimensional code in a $q-1$-dimensional space. Find the minimal Hamming weight of a non-zero element of the code.