ASSIGNMENT 3 - MATH 251, WINTER 2007

Submit by Monday, February 5, 12:00

1. Let \( T : V \rightarrow V \) be a nilpotent linear operator. Prove that if \( n = \dim(V) \) then \( T^n \equiv 0 \). Show that for every \( n \geq 2 \) there exists a vector space \( V \) of dimension \( n \) and a nilpotent linear operator \( T : V \rightarrow V \) such that \( T^{n-1} \neq 0 \).

2. (a) Find a linear map \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) whose image is generated by \((1, 2, 3)\) and \((3, 2, 1)\). Here ‘find’ means represent by a matrix with respect to the standard basis.
   (b) Find a linear map \( T : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \) whose kernel is generated by \((1, 2, 3, 4), (0, 1, 0, 1)\).

3. Let \( W = \{ (x, y, z, w) : x + y + z + w = 0, x - y + 2z - w = 0 \} \), \( U_1 = \{ (x, x, x, x) : x \in \mathbb{R} \} \) be subspaces of \( \mathbb{R}^4 \). Find a subspace \( U \supset U_1 \) such that \( \mathbb{R}^4 = U \oplus W \). Let \( T \) be the projection of \( \mathbb{R}^4 \) on \( U \) along \( W \). Write the matrix representing \( T \) with respect to the standard basis.

4. Let \( V \) and \( W \) be any two finite dimensional vector spaces over a field \( F \). Let \( T : V \rightarrow W \) be a linear map. Prove that there are bases of \( V \) and \( W \) such that with respect to those bases \( T \) is represented by a matrix composed of 0’s and 1’s only. If \( T \) is an isomorphism, prove that with respect to suitable bases it is represented by the identity matrix.

5. Let \( V \) be a vector space of dimension \( n \) and let \( W \) be a vector space of dimension \( m \), both over the same field \( F \). Prove that \( \text{Hom}(V, W) \cong M_{m \times n}(F) \) as vector spaces over \( F \). Here \( M_{m \times n}(F) \) stands for matrices with \( n \) columns and \( m \) rows with entries in \( F \).

6. Consider the transformation that rotates the plane \( \mathbb{R}^2 \) by angle \( \theta \) counter-clockwise. Write this transformation as a matrix in the standard basis. Write it also as a matrix with respect to the basis \((1, 1), (1, 0)\).

7. Let \( G \) be a bipartite regular graph, whose set of left vertices is \( L \) and right vertices is \( R \). For a set \( S \subset L \) denote by \( \partial S := \{ r \in R : r \text{ is connected to a vertex in } S \} \). Suppose that \( |L| = n, |R| = 3n/4 \).
   Suppose that \( G \) has the following expansion property. For every \( S \subset L \) such that \( |S| \leq \frac{n}{10d_L} \) we have \( |\partial S| \geq \frac{5d_L}{4}|S| \). Prove that for every such \( S \) there is a vertex \( r_S \) (many, in fact) such that \( r_S \) is a neighbor of exactly one element of \( S \).
   Consider now the linear code defined by the “half adjacency matrix \( M \)” whose columns are indexed by the elements of \( L \) and rows by the elements of \( R \), having 1 as an entry if the corresponding vertices are connected and 0 otherwise. (Refer to the previous assignment). Prove that if \( x \) is a non-zero vector in the code then \( x \) has more than \( \frac{n}{10d_L} \) non-zero coordinates. Conclude that we get a code where the distance between any two code words is at least \( \frac{n}{10d_L} \) and whose rate is at least \( 1/4 \).