ASSIGNMENT 3 - MATH 251, WINTER 2007

Submit by Monday, February 5, 12:00

1. Let $T : V \to V$ be a nilpotent linear operator. Prove that if $n = \dim(V)$ then $T^n \equiv 0$. Show that for every $n \geq 2$ there exists a vector space V of dimension n and a nilpotent linear operator $T : V \to V$ such that $T^{n-1} \not\equiv 0$.

2. (a) Find a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose image is generated by $(1, 2, 3)$ and $(3, 2, 1)$. Here 'find' means represent by a matrix with respect to the standard basis.
(b) Find a linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ whose kernel is generated by $(1, 2, 3, 4), (0, 1, 0, 1)$.

3. Let $W = \{(x, y, z, w) : x + y + z + w = 0, x - y + 2z - w = 0\}$, $U_1 = \{(x, x, x, x) : x \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^4$. Find a subspace $U \supset U_1$ such that $\mathbb{R}^4 = U \oplus W$. Let T be the projection of $\mathbb{R}^4$ on U along W. Write the matrix representing T with respect to the standard basis.

4. Let V and W be any two finite dimensional vector spaces over a field $\mathbb{F}$. Let $T : V \to W$ be a linear map. Prove that there are bases of V and W such that with respect to those bases T is represented by a matrix composed of 0's and 1's only. If T is an isomorphism, prove that with respect to suitable bases it is represented by the identity matrix.

5. Let V be a vector space of dimension n and let W be a vector space of dimension m, both over the same field $\mathbb{F}$. Prove that $\mathrm{Hom}(V, W) \cong M_{m\times n}(\mathbb{F})$ as vector spaces over $\mathbb{F}$. Here $M_{m\times n}(\mathbb{F})$ stands for matrices with n columns and m rows with entries in $\mathbb{F}$.

6. Consider the transformation that rotates the plane $\mathbb{R}^2$ by angle θ counter-clockwise. Write this transformation as a matrix in the standard basis. Write it also as a matrix with respect to the basis $(1, 1), (1, 0)$.

7. Let G be a bipartite regular graph, whose set of left vertices is L and right vertices is R. For a set $S \subset L$ denote by $\partial S := \{r \in R : r \text{ is connected to a vertex in } S\}$. Suppose that $|L| = n, |R| = 3n/4$.

Suppose that G has the following expansion property. For every $S \subset L$ such that $|S| \leq \frac{n}{10d_L}$ we have $|\partial S| \geq \frac{5d_L}{8}|S|$. Prove that for every such S there is a vertex r_S (many, in fact) such that r_S is a neighbor of exactly one element of S.

Consider now the linear code defined by the "half adjacency matrix M" whose columns are indexed by the elements of L and rows by the elements of R, having 1 as an entry if the corresponding vertices are connected and 0 otherwise. (Refer to the previous assignment). Prove that if x is a non-zero vector in the code then x has more than $\frac{n}{10d_L}$ non-zero coordinates. Conclude that we get a code where the distance between any two code words is at least $\frac{n}{10d_L}$ and whose rate is at least $1/4$.