ASSIGNMENT 2 - MATH 251, WINTER 2007

Submit by Monday, January 29, 12:00

1. Let $\mathcal{B} = \{(1,1),(1,5)\}$ and $\mathcal{C} = \{(2,1),(1,-1)\}$ be bases of $\mathbb{R}^2$. Find the change of basis matrices ${}_{\mathcal{B}}M_{\mathcal{C}}$ and ${}_{\mathcal{C}}M_{\mathcal{B}}$ between the bases $\mathcal{B}$ and $\mathcal{C}$. Let $v = \left(\begin{smallmatrix} 8 \\ 28 \end{smallmatrix}\right)$ with respect to the standard basis. Find $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{C}}$.

2. Which of the following functions is a linear map? (provide proof):

(1) $T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x,y) = (3x - 2y, x + y)$.
(2) $T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x,y) = (x^2 - y, x + y + 1)$.
(3) $T : \mathbb{R}[x]_3 \to \mathbb{R}^2, \quad T(f(x)) = (f(1), f'(1))$.
(4) $T : \mathbb{R}[x]_3 \to \mathbb{R}[x]_4, \quad T(f(x)) = xf(x) + f(1)$.

In each case where T is a linear map, find its kernel.

3. Let $T : V \to W$ be a surjective linear map. Prove that if $\dim(V) = \dim(W)$ then T is an isomorphism.

4. Prove the following Proposition.

Proposition 0.1. *Let V and W be vector spaces over $\mathbb{F}$. Let $B = \{b_1, \dots, b_n\}$ be a basis for V and let $t_1, \dots, t_n$ be any elements of W. There is a unique linear map*

$$T : V \to W,$$

such that

$$T(b_i) = t_i, \qquad i = 1, \dots, n.$$

5. Prove the following:

Lemma 0.2. *Let V, W be vector spaces over $\mathbb{F}$. Let*

$$\mathrm{Hom}(V,W) = \{T : V \to W : f \text{ is a linear map}\}.$$

Then $\mathrm{Hom}(V,W)$ is a vector space in its own right where we define for two linear transformations S, T and scalar α the linear transformations $S + T, \alpha S$ as follows:

$$(S+T)(v) = S(v) + T(v), \qquad (\alpha S)(v) = \alpha S(v).$$

In addition, if $T : V \to W$ and $R : W \to U$ are linear maps, where U is a third vector space over $\mathbb{F}$, then

$$R \circ T : V \to U$$

is a linear map.

6. Consider the following subspaces of $\mathbb{R}^3$:

$$U = \{(x, y, z) \in \mathbb{R}^3 : x + y - z = 0, x - 2y + z = 0\}, \quad W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}.$$

Answer the following questions.

(1) Find the dimensions of U and W.
(2) Is there a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $U = \mathrm{Ker}(T), W = \mathrm{Im}(T)$?
(3) In case T exists, is it unique?

7. **Graphs.** A graph G consists of a finite set of vertices $V(G)$ and a set of edges $E(G) \subset V(G) \times V(G)$. Thus, a vertex u is connected to a vertex v precisely if $(u, v) \in E(G)$. We assume that the graph is non-oriented, meaning $(u, v) \in E(G) \Leftrightarrow (v, u) \in E(G)$. Note that by our definition there is at most one edge between any two vertices. We shall also assume, for simplicity, that the graph has no loops. Namely, $(v, v) \notin E(G)$ for all $v \in V(G)$.

Suppose that the vertices are labeled $1, 2, \dots, n$. We may encode the graph by a symmetric matrix A, called the adjacency matrix of the graph. We define

$$A = (a_{ij}), \quad a_{ij} = \begin{cases} 1 & (i, j) \in E(G) \\ 0 & \text{else.} \end{cases}$$

A. Write the adjacency matrix of the graph whose vertices are $\{0, \dots, 6\}$ *where any* i *is connected to* $i + 1 \pmod 7$ *and* $i + 2 \pmod 7$*. Draw the graph as well.*

Let k be an integer. A graph is called k-regular if from every vertex there are exactly k edges.

B. Show that the above graph is 4*-regular. Prove that a graph is* k*-regular if and only if the sum of the coefficients of every row (every column) of its adjacency matrix is* k*.*

A graph is called bipartite if we can divide its vertices into two disjoint sets L and R such that vertices from L are only connected to vertices in R and vertices in R are only connected to vertices in L. We shall say that a bipartite graph is regular if for some d_L, d_R, every vertex in L connects to exactly d_L vertices in R and any vertex in R connects to exactly d_R vertices in L.

C. Prove that in this case $|L| \cdot d_L = |R| \cdot d_R$*. Is our example above bipartite?*

D. For a bipartite graph G*, with* $L = \{\ell_1, \dots, \ell_n\}, R = \{r_1, \dots, r_m\}$*, define an* $m \times n$ *matrix* M *whose* ij *entry is* 1 *if* r_i *is connected to* ℓ_j *and zero otherwise. How can we write the adjacency matrix of* G *in terms of* M*? How can we determine in terms of* M *whether the graph is regular?*

If $C \subset \mathbb{F}^n$ is a linear code, we define its rate to be

$$r(C) = \dim(C)/n.$$

This magnitude measures the efficiency of the code. Note that $0 \le r(C) \le 1$.

Now take the matrix M we have just constructed from a bipartite graph G and consider it as a linear map

$$T : \mathbb{F}_2^n \to \mathbb{F}_2^m, \qquad T(v) = Mv,$$

(we think of the vectors as column vectors). Define a linear code by taking the kernel of this linear transformation. For future reference we denote this code by C_G,

$$C_G = \{v \in \mathbb{F}_2^n; Mv = 0\}.$$

E. Prove that the rate of C_G *is at least* $\frac{n-m}{n}$.

F. Consider the following example. Let $R = \{1, 2, \ldots, n\}$ *and let* L *the set whose elements are subsets of order* k *of* R*. Thus,* L *has* $\binom{n}{k}$*-elements. An element* a *of* R *is connected to an element* S *of* L *if and only if* $a \in S$*. Show that this gives a regular bipartite graph. Find the rate and distance of the linear code for* $n = 4$ *and* $k = 2$*. The general case (i.e., any* n *and* k*) seems a challenging problem. Let me know if you can solve it!*