ASSIGNMENT 11 - MATH 251, WINTER 2007

Do NOT submit

1. It is known that a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ has a
   - **maximum** at a point P if $\partial f/\partial x = \partial f/\partial y = 0$ at P and the 2×2 symmetric matrix
   $$-\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$
   is positive definite;
   - **minimum** at a point P if $\partial f/\partial x = \partial f/\partial y = 0$ at P and the 2×2 symmetric matrix
   $$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$
   is positive definite;
   - **saddle point** at P if the 2×2 symmetric matrix
   $$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$
   has one negative eigenvalue and one positive eigenvalue.

If P is either a maximum, minimum or saddle point, we call it a simple critical point. Determine the nature of the simple critical point of the following functions at the origin $(0, 0)$

$$f(x, y) = 2x^2 + 6xy + y^2, \quad f(x, y) = x \sin(x) - \cos(y) - xy.$$

(You may view the graphs and rotate them in Maple using
  plot3d(2*x^ 2+6*x*y+y^ 2, x= -4..4, y = -4..4);
  plot3d(x*sin(x) - cos(y) -x*y, x= -4..4, y = -4..4, numpoints=3000);   ).

Remark. This criterion can be generalized to functions $f : \mathbb{R}^n \to \mathbb{R}$. If all the first partials vanish at a point P and the matrix of mixed derivatives is positive definite (resp. negative definite) at P, then the function has a minimum (resp. maximum) at P.

2. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal, where
$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

3. Prove that the determinant of a unitary matrix has absolute value 1.

4. Prove that the following are equivalent for a matrix $A \in M_n(\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$):
   (1) A is unitary.
   (2) A preserves inner products: $\langle v, w \rangle = \langle Av, Aw \rangle$ for all $v, w \in \mathbb{F}^n$;
   (3) A preserves lengths: $\|Av\| = \|v\|$ for all $v \in \mathbb{C}^n$.

5. Prove that the unitary $n \times n$ matrices form a group under matrix multiplication and the orthogonal matrices are a subgroup, they are denoted respectively $U_n(\mathbb{C}), O_n(\mathbb{R})$. Describe explicitly those groups for $n = 1, 2$.

6. Prove that the following are equivalent for a matrix $A \in M_n(\mathbb{C})$:
   (1) $A = A^*$ and is positive definite (as defined in class);
   (2) $A = S^*S$ for some non-singular matrix $S \in M_n(\mathbb{C})$;
   (3) $A = T^2$ for some self-adjoint non-singular operator T.

7. Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a normal operator. Prove the following:
   (1) T is self-adjoint if and only its eigenvalues are real;
   (2) T is unitary if and only if its eigenvalues have absolute value one;
   (3) T is a product US where U is unitary, S is self-adjoint and $US = SU$.