Solutions to Midterm Exam, MATH 133 - Vectors, Matrices and Geometry

Date: Monday, May 17, 2004.      Time: 14:00 - 16:00.

1. Let \( u, v \) be vectors in \( \mathbb{R}^n \). Prove that \( u + v \) and \( u - v \) are orthogonal if and only if \( \|u\| = \|v\| \).

Proof. We have \((u+v) \perp (u-v)\) iff \((u+v) \cdot (u-v) = 0\). But \((u+v) \cdot (u-v) = u \cdot u + v \cdot v - u \cdot v - v \cdot v = \|u\|^2 - \|v\|^2\). This is equal to zero iff \( \|u\|^2 = \|v\|^2 \), iff \( \|u\| = \|v\| \) (as norms are non-negative). □

2. Let \( \mathcal{B} = \{v_1, \ldots, v_n\} \) be a basis of \( \mathbb{R}^n \). Prove that the following properties hold:
   
   (1) Every vector in \( \mathbb{R}^n \) has a \textit{unique} expression as a linear combination of the basis vectors \( v_1, \ldots, v_n \).

   (2) \( \mathcal{B} \) is a maximally independent set, namely, for any vector \( w \in \mathbb{R}^n \) the set \( \mathcal{B}' = \{v_1, \ldots, v_n, w\} \) is linearly dependent.

Proof. (1) Since a basis is a spanning set, every vector \( w \in \mathbb{R}^n \) has some expression as \( w = a_1v_1 + \cdots + a_nv_n \). Suppose that also \( w = b_1v_1 + \cdots + b_nv_n \). Then, subtracting the two equations, we find that \( 0 = w - w = (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n \). Since a basis is a linearly independent set, this implies \( a_i - b_i = 0 \) for all \( i \) and so \( a_i = b_i \) for all \( i \).

(2) By the first part, \( w = a_1v_1 + \cdots + a_nv_n \) for some \( a_i \) and so \( a_1v_1 + a_2v_2 + \cdots + a_nv_n - 1 \cdot w = 0 \). This is a non-trivial linear combination that shows that the vectors \( \{v_1, \ldots, v_n, w\} \) are linearly equivalent. □

3. Find a basis for the solutions to the homogeneous system

\[
\begin{align*}
x_1 + x_2 + 2x_3 - x_4 - 4x_5 &= 0 \\
-x_1 + x_3 + 2x_4 - 5x_5 &= 0 \\
2x_1 - x_2 - 5x_3 - 4x_4 + 15x_5 &= 0 \\
2x_1 + 3x_2 + 7x_3 - x_4 - 17x_5 &= 0
\end{align*}
\]

Solution: The system corresponds to the matrix

\[
A = \begin{pmatrix}
1 & 1 & 2 & -1 & -4 \\
-1 & 0 & 1 & 2 & -5 \\
2 & -1 & -5 & -4 & 15 \\
2 & 3 & 7 & -1 & -17
\end{pmatrix}.
\]
This matrix has reduced echelon form
\[
\begin{pmatrix}
1 & 0 & -1 & 0 & -3 \\
0 & 1 & 3 & 0 & -5 \\
0 & 0 & 0 & 1 & -4 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The free variables are \( x_3 \) and \( x_5 \). Setting them equal to \((1,0)\) and \((0,1)\) respectively, we find the basis \( \{u_1, u_2\} \) with
\[
\begin{align*}
u_1 &= (1,-3,1,0,0), \\
u_2 &= (3,5,0,4,1)
\end{align*}
\]

4. Write the parametric form of the line passing through the point \( R = (3,9,3) \) and perpendicular to the plane \( P \) given by \( 7x - y - 4z = -11 \) as
\[
\begin{align*}
x &= \_\_\_ + \_\_\_ \cdot t \\
y &= \_\_\_ + \_\_\_ \cdot t \\
z &= \_\_\_ + \_\_\_ \cdot t
\end{align*}
\]
Find the distance between the point \( R \) and the plane \( P \).

**Solution:** The line is given by \( \{R + tn : t \in \mathbb{R}\} \), where \( n \) is any vector normal to the plane \( P \). Such a vector is in fact supplied by the equation. We may take \( n = (7,-1,-4) \) and we conclude that the line is \( \{(3+7t,9-t,3-4t) : t \in \mathbb{R}\} \). That is,
\[
\begin{align*}
x &= 3 + 7 \cdot t, \\
y &= 9 - t, \\
z &= 3 - 4 \cdot t.
\end{align*}
\]
The line intersects the plane at a point \((x,y,z)\) such that \( 7(3+7t) - (9-t) - 4(3-4t) = -11 \). That is, \(66 \cdot t = -11\). Thus, \( t = -1/6 \) and the point is \( R - \frac{1}{6}(7,-1,4) \). The distance of \( R \) from the plane is the distance between the point \( R \) and the point \( R - \frac{1}{6}(7,-1,4) \), namely, the norm of \( \frac{1}{6}(7,-1,4) \) which is \( \sqrt{\frac{11}{6}} \).

5. Find the inverse of the matrix \( A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & -3 \end{pmatrix} \). Find a matrix \( X \) such that \( AXA = A^2 + A \).

**Solution:** The inverse can be found by row-reduction (the Gauss-Jordan method for finding the inverse). One finds that
\[
A^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -3 & -4 \\ 0 & 1 & 1 \end{pmatrix}.
\]
The equation \( AXA = A^2 + A \) is equivalent to \( XA = A + I_3 \) (multiply by \( A^{-1} \) on the left) and so to \( X = I_3 + A^{-1} \) (multiply by \( A^{-1} \) on the right). Thus,
\[
X = \begin{pmatrix} 2 & -1 & -1 \\ 0 & -2 & -4 \\ 0 & 1 & 2 \end{pmatrix}.
\]
6. Let \( A = \begin{pmatrix} -1 & 1 & 2 & -2 \\ 2 & 1 & -10 & 0 \\ 2 & 0 & -8 & 1 \\ 1 & 2 & -8 & 0 \end{pmatrix} \). Find the dimension of the row space of \( A \) and a basis for it.

**Solution:** We row-reduce the matrix \( A \) to find that its REF is
\[
\begin{pmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

It follows that the dimension of the row space is 3 and the vectors \((1, 0, -4, 0), (0, 1, -2, 0), (0, 0, 0, 1)\) are a basis for it.