Solutions to Midterm Exam, MATH 133 - Vectors, Matrices and Geometry

Date: Monday, May 17, 2004. **Time:** 14:00 - 16:00.

1. Let u, v be vectors in $\mathbb{R}^n$. Prove that $u+v$ and $u-v$ are orthogonal if and only if $\|u\| = \|v\|$.

Proof. We have $(u+v) \perp (u-v)$ iff $(u+v)\cdot(u-v) = 0$. But $(u+v)\cdot(u-v) = u\cdot u + v\cdot u - u\cdot v - v\cdot v = \|u\|^2 - \|v\|^2$. This is equal to zero iff $\|u\|^2 = \|v\|^2$, iff $\|u\| = \|v\|$ (as norms are non-negative). $\square$

2. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of $\mathbb{R}^n$. Prove that the following properties hold:

(1) Every vector in $\mathbb{R}^n$ has a *unique* expression as a linear combination of the basis vectors $v_1, \dots, v_n$.

(2) $\mathcal{B}$ is a maximally independent set, namely, for any vector $w \in \mathbb{R}^n$ the set $\mathcal{B}' = \{v_1, \dots, v_n, w\}$ is linearly dependent.

Proof. (1) Since a basis is a spanning set, every vector $w \in \mathbb{R}^n$ has some expression as $w = a_1v_1 + \cdots + a_nv_n$. Suppose that also $w = b_1v_1 + \cdots + b_nv_n$. Then, subtracting the two equations, we find that $0 = w - w = (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n$. Since a basis is a linearly independent set, this implies $a_i - b_i = 0$ for all i and so $a_i = b_i$ for all i.

(2) By the first part, $w = a_1v_1 + \cdots + a_nv_n$ for some a_i and so $a_1v_1 + a_2v_2 + \cdots + a_nv_n - 1\cdot w = 0$. This is a non-trivial linear combination that shows that the vectors $\{v_1, \dots, v_n, w\}$ are linearly equivalent.

$\square$

3. Find a basis for the solutions to the homogeneous system

$$\begin{aligned}
&x_1 + x_2 + 2x_3 - x_4 - 4x_5 = 0\\
&-x_1 + x_3 + 2x_4 - 5x_5 = 0\\
&2x_1 - x_2 - 5x_3 - 4x_4 + 15x_5 = 0\\
&2x_1 + 3x_2 + 7x_3 - x_4 - 17x_5 = 0
\end{aligned}$$

Solution: The system corresponds to the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & -1 & -4 \\ -1 & 0 & 1 & 2 & -5 \\ 2 & -1 & -5 & -4 & 15 \\ 2 & 3 & 7 & -1 & -17 \end{pmatrix}.$$

This matrix has reduced echelon form

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -3 \\ 0 & 1 & 3 & 0 & -5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The free variables are x_3 and x_5. Setting them equal to $(1,0)$ and $(0,1)$ respectively, we find the basis $\{u_1, u_2\}$ with

$u_1 = (1, -3, 1, 0, 0)$,

$u_2 = (3, 5, 0, 4, 1)$

4. Write the parametric form of the line passing through the point $R = (3, 9, 3)$ and perpendicular to the plane $\mathscr{P}$ given by $7x - y - 4z = -11$ as

$x =$ ____ $+$ ____ $\cdot t$

$y =$ ____ $+$ ____ $\cdot t$

$z =$ ____ $+$ ____ $\cdot t$

Find the distance between the point R and the plane $\mathscr{P}$.

Solution: The line is given by $\{R + tn : t \in \mathbb{R}\}$, where n is any vector normal to the plane $\mathscr{P}$. Such a vector is in fact supplied by the equation. We may take $n = (7, -1, -4)$ and we conclude that the line is $\{(3 + 7t, 9 - t, 3 - 4t) : t \in \mathbb{R}\}$. That is,

$x = 3 + 7 \cdot t$,

$y = 9 - t$,

$z = 3 - 4 \cdot t$.

The line intersects the plane at a point (x, y, z) such that $7(3+7\cdot t) - (9-t) - 4(3-4\cdot t) = -11$. That is, $66 \cdot t = -11$. Thus, $t = -1/6$ and the point is $R - \frac{1}{6}(7, -1, 4)$. The distance of R from the plane is the distance between the point R and the point $R - \frac{1}{6}(7, -1, 4)$, namely, the norm of $\frac{1}{6}(7, -1, 4)$ which is $\sqrt{\frac{11}{6}}$.

5. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & -1 & -3 \end{pmatrix}$. Find a matrix X such that $AXA = A^2 + A$.

Solution: The inverse can be found by row-reduction (the Gauss-Jordan method for finding the inverse). One finds that

$$A^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -3 & -4 \\ 0 & 1 & 1 \end{pmatrix}.$$

The equation $AXA = A^2 + A$ is equivalent to $XA = A + I_3$ (multiply by A^{-1} on the left) and so to $X = I_3 + A^{-1}$ (multiply by A^{-1} on the right). Thus,

$$X = \begin{pmatrix} 2 & -1 & -1 \\ 0 & -2 & -4 \\ 0 & 1 & 2 \end{pmatrix}.$$

6. Let $A = \begin{pmatrix} -1 & 1 & 2 & -2 \\ 2 & 1 & -10 & 0 \\ 2 & 0 & -8 & 1 \\ 1 & 2 & -8 & 0 \end{pmatrix}$. Find the dimension of the row space of A and a basis for it.

Solution: We row-reduce the matrix A to find that its REF is

$$\begin{pmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that the dimension of the row space is 3 and the vectors $(1, 0, -4, 0)$, $(0, 1, -2, 0)$, $(0, 0, 0, 1)$ are a basis for it.