Solutions of systems of linear equations

The system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

corresponds to the matrix $(A|b)$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}| & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n}| & b_2 \\ & & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn}| & b_m \end{pmatrix}$$

One finds the solutions by reducing the matrix A to a reduced echelon form, getting a system $(M|c)$. If $(t_1, \ldots, t_n)$ is any solution to this system then all the solutions have the following form

$$\{(t_1, \ldots, t_n) + (s_1, \ldots, s_n) | (s_1, \ldots, s_n) \text{ a solution of the homog. system } (M|0)\}$$

For a matrix M in reduced echelon form we have the notion of free variables $x_{i_1}, \ldots, x_{i_r}$. The indices $i_1, \ldots, i_r$ are the indices in which there is no echelon. If we choose any values $s_{i_1}, \ldots, s_{i_r}$ for the free variables then there is a unique way to choose values for the rest of the variables so that we get a solution to the homogenous system. In particular, we can define the following solutions:

u_1 with $s_{i_1} = 1, s_{i_2} = 0, \ldots, s_{i_r} = 0$;
u_2 with $s_{i_1} = 0, s_{i_2} = 1, s_{i_3} = 0, \ldots, s_{i_r} = 0$;
etc...
u_r with $s_{i_1} = 0, \ldots, s_{i_{r-1}} = 0, s_{i_r} = 1$.

We also remark that the solution $(t_1, \ldots, t_n)$ to the non-homogenous system can be found by putting all the free variables equal to 0. The rest of the variables are then getting the values $c_1, \ldots, c_m$.

The solutions to the homogenous system $(A|0)$ or $(M|0)$ (these are equivalent systems having the same solutions) can be written as follows: they are of the form

$$c_1 u_1 + \cdots + c_r u_r,$$

where the $c_1, \ldots, c_r$ are scalars and the vectors $u_1, \ldots, u_r$ are as above. In fact *the vectors $u_1, \ldots, u_r$ are a basis to the subspace of all solutions to the homogenous system.*

For example, consider the system

$$\begin{pmatrix} 0 & |\underline{1} & \underline{2} & 0 & 5 & 0| & 1 \\ 0 & 0 & 0 & |\underline{1} & \underline{3} & 0| & 2 \\ 0 & 0 & 0 & 0 & 0 & |\underline{1}| & 3 \end{pmatrix}$$

The free variables are x_1, x_3, x_5. We put $x_1 = 0, x_3 = 0, x_5 = 0$ and find the solution $(\mathbf{0}, 1, \mathbf{0}, 2, \mathbf{0}, 3)$ to the non-homogenous system. The solutions to the homogenous system are spanned by the solutions u_1, u_2, u_3 obtained by choosing $\{x_1 = 1, x_3 = 0, x_5 = 0\}$, $\{x_1 = 0, x_3 = 1, x_5 = 0\}$ and $\{x_1 = 0, x_3 = 0, x_5 = 1\}$, respectively. That is, the solutions

$u_1 = (\mathbf{1}, 0, \mathbf{0}, 0, \mathbf{0}, 0)$,
$u_2 = (\mathbf{0}, -2, \mathbf{1}, 0, \mathbf{0}, 0)$,
$u_3 = (\mathbf{0}, -5, \mathbf{0}, -3, \mathbf{1}, 0)$.

For another example, consider the homogenous system

$$\begin{pmatrix} 0 & |\underline{1} & \underline{9} & \underline{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & |\underline{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & |\underline{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & |\underline{1} & \underline{-1} \end{pmatrix}$$

The free variables are x_1, x_3, x_4, x_8. We can find the basis vectors $u_1, \ldots, u_4$ using the table (we fill first the first 4 columns and then back-substitute to find the rest).

	x_1	x_3	x_4	x_8	x_2	x_5	x_6	x_7
u_1	1	0	0	0	0	0	0	0
u_2	0	1	0	0	-9	0	0	0
u_3	0	0	1	0	-2	0	0	0
u_4	0	0	0	1	-1	-1	0	1

That is

$u_1 = (\mathbf{1}, 0, \mathbf{0}, \mathbf{0}, 0, 0, 0, \mathbf{0})$

$u_2 = (\mathbf{0}, -9, \mathbf{1}, \mathbf{0}, 0, 0, 0, \mathbf{0})$

$u_3 = (\mathbf{0}, -2, \mathbf{0}, \mathbf{1}, 0, 0, 0, \mathbf{0})$

$u_0 = (\mathbf{0}, -1, \mathbf{0}, \mathbf{0}, -1, 0, 1, 1)$