MATH 133 - Vectors, Matrices and Geometry
Final Examination

Time and location: 9:00 - 12:00, Friday, May 28, 2004. LEA 14 & 15.
Instructions: Answer at least 7 questions. The maximal grade possible is 105. Write carefully and in an orderly fashion all the details of the calculations. *Books, Notes, formula sheets and* <u>*calculators*</u> *are not allowed.*

1. Let A be an $n \times n$ diagonalizable matrix. Prove that there is a basis consisting of eigenvectors of A.

2. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Let v_1, v_2, v_3 be vectors in $\mathbb{R}^n$ such that $T(v_1), T(v_2), T(v_3)$ are linearly independent. Prove that v_1, v_2, v_3 are linearly independent.

3.

(1) Find the equation of the plane $\mathscr{P}$ in $\mathbb{R}^3$ passing through the points $P = (1, 1, 1), Q = (1, 0, 1), R = (-1, 0, 0)$. Namely, find a, b, c, d such that the plane is given in the form $ax + by + cz = d$.
(2) Find a vector of length 1 which is normal to $\mathscr{P}$.
(3) Find a point on $\mathscr{P}$ whose distance from P and Q is the same.

4. Let W be the subspace of $\mathbb{R}^5$ which is the span of the following vectors:

$$(1, 0, 1, 1, 0), (1, 2, -1, 1, 2), (4, 2, 2, 4, 2), (5, 14, -9, 5, 14).$$

(1) Find the dimension of W.
(2) Find an orthogonal basis for W.
(3) Find the orthogonal projection of the vector $(1, 0, 0, 0, 1)$ on W.

5. Let $A = \begin{pmatrix} 2k+1 & k \\ -4 & -1 \end{pmatrix}$. For which values of k is the matrix A diagonalizable? For those values, write a diagonal matrix similar to A.

6. Let $A = \begin{pmatrix} 2 & 3 & -3 \\ 3 & 2 & 3 \\ -3 & 3 & 2 \end{pmatrix}$. The number 5 is an eigenvalue of A.

(1) Find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$.

(2) Write the quadratic form determined by A. Find a change of variables $X = PY$ so that that the quadratic form has the form $ay_1^2 + by_2^2 + cy_3^2$. What are a, b, c?

7. For which k does the following system have: (i) no solutions? (ii) a unique solution? (iii) infinitely many solutions? (Note that it is possible that not all cases occur.)

$$kx - 3y = 1, \quad 6x + 9y = -3.$$

8. Find the standard matrix of each of the following linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$:

(1) T_1, the projection onto the line $x + 3y = 0$.

(2) T_2, the reflection in the line $2x + y = 0$.

In addition, write the matrix of T_1T_2.

Good Luck!!