Diagonalization algorithms

Diagonalization of any matrix

Goal: Given an $n \times n$ matrix A, to find an invertible matrix P (if such exists[1]) so that $P^{-1}AP = D$ is diagonal.

(1) Calculate the characteristic polynomial $f(x) = \det(A - xI_n)$ of A.

(2) Write $f(x) = (-1)^n (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\cdots(x-\lambda_r)^{m_r}$. Note that $m_1+m_2+\cdots+m_r = n$; the λ_i are the eigenvalues of A.

(3) Calculate the eigenspace $E_{\lambda_i} = \{v : (A - \lambda_i I_n)v = 0\}$ for every eigenvalue λ_i.

- If for some eigenvalue λ_i we have $\dim(E_{\lambda_i}) < m_i$ then the algorithm stops. The matrix cannot be diagonalized.
- If for every eigenvalue λ_i we have $\dim(E_{\lambda_i}) = m_i$ then the algorithm continues. The matrix can be diagonalized.

(4) Calculate a basis $\mathscr{B}_i$ for every eigenspace E_{λ_i}. Let $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \cdots \cup \mathscr{B}_r$. Then each $\mathscr{B}_i$ consists of m_i vectors and $\mathscr{B} = \{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$.

(5) Let $P = (v_1|v_2|\cdots|v_n)$. Then P is invertible and $P^{-1}AP$ is the diagonal matrix with the following blocks

$$\begin{pmatrix} m_1\boxed{\begin{matrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_1\end{matrix}} & & & \\ & m_2\boxed{\begin{matrix}\lambda_2 & & \\ & \ddots & \\ & & \lambda_2\end{matrix}} & & \\ & & \ddots & \\ & & & m_r\boxed{\begin{matrix}\lambda_r & & \\ & \ddots & \\ & & \lambda_r\end{matrix}} \end{pmatrix}$$

[1] Such a matrix does not always exist, e.g., $A = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ is not diagonalizable.

Orthogonal diagonalization of any real symmetric matrix

Goal: Given an $n \times n$ real symmetric matrix A, to find[2] *an orthogonal real matrix P so that $P^T AP = D$ is diagonal.*

(1) Calculate the characteristic polynomial $f(x) = \det(A - xI_n)$ of A.

(2) Write $f(x) = (-1)^n (x-\lambda_1)^{m_1} (x-\lambda_2)^{m_2} \cdots (x-\lambda_r)^{m_r}$. Note that $m_1 + m_2 + \cdots + m_r = n$; the λ_i are the eigenvalues of A are real numbers.

(3) Calculate the eigenspace $E_{\lambda_i} = \{v : (A - \lambda_i I_n)v = 0\}$ for every eigenvalue λ_i; for every eigenvalue λ_i we have $\dim(E_{\lambda_i}) = m_i$.

(4) Calculate a basis $\mathscr{C}_i$ for every eigenspace E_{λ_i}. Using Gram-Schmidt calculate from $\mathscr{C}_i$ an orthonormal basis $\mathscr{B}_i$ for E_{λ_i}. Let $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \cdots \cup \mathscr{B}_r$. Then each $\mathscr{B}_i$ consists of m_i vectors and $\mathscr{B} = \{v_1, \dots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

(5) Let $P = (v_1 | v_2 | \cdots | v_n)$. Then P is an orthogonal matrix and $P^T AP$ is the diagonal matrix with blocks

$$\begin{pmatrix} m_1\boxed{\begin{matrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_1\end{matrix}} & & & \\ & m_2\boxed{\begin{matrix}\lambda_2 & & \\ & \ddots & \\ & & \lambda_2\end{matrix}} & & \\ & & \ddots & \\ & & & m_r\boxed{\begin{matrix}\lambda_r & & \\ & \ddots & \\ & & \lambda_r\end{matrix}} \end{pmatrix}$$

[2]Such a matrix always exists.