Diagonalization algorithms

Diagonalization of any matrix

Goal: Given an $n \times n$ matrix $A$, to find an invertible matrix $P$ (if such exists\(^1\)) so that $P^{-1}AP = D$ is diagonal.

(1) Calculate the characteristic polynomial $f(x) = \det(A - xI_n)$ of $A$.

(2) Write $f(x) = (-1)^n(x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \cdots (x - \lambda_r)^{m_r}$. Note that $m_1 + m_2 + \cdots + m_r = n$; the $\lambda_i$ are the eigenvalues of $A$.

(3) Calculate the eigenspace $E_{\lambda_i} = \{v : (A - \lambda_i I_n)v = 0\}$ for every eigenvalue $\lambda_i$.
   - If for some eigenvalue $\lambda_i$ we have $\dim(E_{\lambda_i}) < m_i$ then the algorithm stops. The matrix cannot be diagonalized.
   - If for every eigenvalue $\lambda_i$ we have $\dim(E_{\lambda_i}) = m_i$ then the algorithm continues. The matrix can be diagonalized.

(4) Calculate a basis $B_i$ for every eigenspace $E_{\lambda_i}$. Let $B = B_1 \cup B_2 \cup \cdots \cup B_r$. Then each $B_i$ consists of $m_i$ vectors and $B = \{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$.

(5) Let $P = (v_1|v_2|\cdots|v_n)$. Then $P$ is invertible and $P^{-1}AP$ is the diagonal matrix with the following blocks

\[
\begin{pmatrix}
\lambda_1 \\
m_1 \\
\vdots \\
\lambda_1
\end{pmatrix}
\begin{pmatrix}
\lambda_2 \\
m_2 \\
\vdots \\
\lambda_2
\end{pmatrix}
\cdots
\begin{pmatrix}
\lambda_r \\
m_r \\
\vdots \\
\lambda_r
\end{pmatrix}
\]

\(^1\)Such a matrix does not always exist, e.g., $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.
Orthogonal diagonalization of any real symmetric matrix

Goal: Given an $n \times n$ real symmetric matrix $A$, to find an orthogonal real matrix $P$ so that $P^T A P = D$ is diagonal.

1. Calculate the characteristic polynomial $f(x) = \det(A - xI_n)$ of $A$.
2. Write $f(x) = (-1)^n(x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \cdots (x - \lambda_r)^{m_r}$. Note that $m_1 + m_2 + \cdots + m_r = n$; the $\lambda_i$ are the eigenvalues of $A$ are real numbers.
3. Calculate the eigenspace $E_{\lambda_i} = \{v : (A - \lambda_i I_n)v = 0\}$ for every eigenvalue $\lambda_i$; for every eigenvalue $\lambda_i$ we have $\dim(E_{\lambda_i}) = m_i$.
4. Calculate a basis $C_i$ for every eigenspace $E_{\lambda_i}$. Using Gram-Schmidt calculate from $C_i$ an orthonormal basis $B_i$ for $E_{\lambda_i}$. Let $\mathcal{B} = B_1 \cup B_2 \cup \cdots \cup B_r$. Then each $B_i$ consists of $m_i$ vectors and $\mathcal{B} = \{v_1, \ldots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$.
5. Let $P = (v_1 | v_2 | \cdots | v_n)$. Then $P$ is an orthogonal matrix and $P^T A P$ is the diagonal matrix with blocks

$$
\begin{pmatrix}
\lambda_1 \\
m_1 \\
\vdots \\
\lambda_1 \\
\end{pmatrix},
\begin{pmatrix}
\lambda_2 \\
m_2 \\
\vdots \\
\lambda_2 \\
\end{pmatrix},
\begin{pmatrix}
\lambda_r \\
m_r \\
\vdots \\
\lambda_r \\
\end{pmatrix}
$$

Such a matrix always exists.