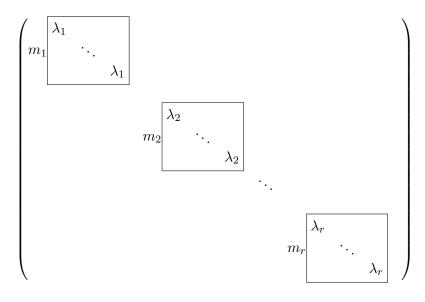
## **Diagonalization algorithms**

## **Diagonalization of any matrix**

Goal: Given an  $n \times n$  matrix A, to find an invertible matrix P (if such exists<sup>1</sup>) so that  $P^{-1}AP = D$  is diagonal.

- (1) Calculate the characteristic polynomial  $f(x) = \det(A xI_n)$  of A.
- (2) Write  $f(x) = (-1)^n (x \lambda_1)^{m_1} (x \lambda_2)^{m_2} \cdots (x \lambda_r)^{m_r}$ . Note that  $m_1 + m_2 + \cdots + m_r = n$ ; the  $\lambda_i$  are the eigenvalues of A.
- (3) Calculate the eigenspace  $E_{\lambda_i} = \{v : (A \lambda_i I_n)v = 0\}$  for every eigenvalue  $\lambda_i$ .
  - If for some eigenvalue  $\lambda_i$  we have  $\dim(E_{\lambda_i}) < m_i$  then the algorithm stops. The matrix cannot be diagonalized.
  - If for every eigenvalue  $\lambda_i$  we have  $\dim(E_{\lambda_i}) = m_i$  then the algorithm continues. The matrix can be diagonalized.
- (4) Calculate a basis  $\mathscr{B}_i$  for every eigenspace  $E_{\lambda_i}$ . Let  $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \cdots \cup \mathscr{B}_r$ . Then each  $\mathscr{B}_i$  consists of  $m_i$  vectors and  $\mathscr{B} = \{v_1, \ldots, v_n\}$  is a basis for  $\mathbb{R}^n$ .
- (5) Let  $P = (v_1 | v_2 | \cdots | v_n)$ . Then P is invertible and  $P^{-1}AP$  is the diagonal matrix with the following blocks

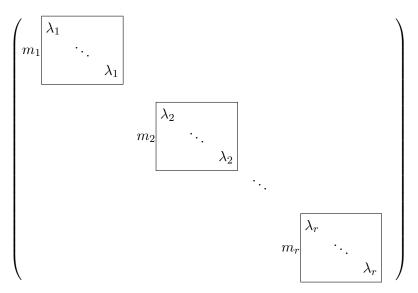


<sup>&</sup>lt;sup>1</sup>Such a matrix does not always exist, e.g.,  $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  is not diagonalizable.

## Orthogonal diagonalization of any real symmetric matrix

Goal: Given an  $n \times n$  real symmetric matrix A, to find<sup>2</sup> an orthogonal real matrix P so that  $P^{T}AP = D$  is diagonal.

- (1) Calculate the characteristic polynomial  $f(x) = \det(A xI_n)$  of A.
- (2) Write  $f(x) = (-1)^n (x \lambda_1)^{m_1} (x \lambda_2)^{m_2} \cdots (x \lambda_r)^{m_r}$ . Note that  $m_1 + m_2 + \cdots + m_r = n$ ; the  $\lambda_i$  are the eigenvalues of A are real numbers.
- (3) Calculate the eigenspace  $E_{\lambda_i} = \{v : (A \lambda_i I_n)v = 0\}$  for every eigenvalue  $\lambda_i$ ; for every eigenvalue  $\lambda_i$  we have dim $(E_{\lambda_i}) = m_i$ .
- (4) Calculate a basis  $\mathscr{C}_i$  for every eigenspace  $E_{\lambda_i}$ . Using Gram-Schmidt calculate from  $\mathscr{C}_i$  an orthonormal basis  $\mathscr{B}_i$  for  $E_{\lambda_i}$ . Let  $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \cdots \cup \mathscr{B}_r$ . Then each  $\mathscr{B}_i$  consists of  $m_i$  vectors and  $\mathscr{B} = \{v_1, \ldots, v_n\}$  is an orthonormal basis for  $\mathbb{R}^n$ .
- (5) Let  $P = (v_1|v_2|\cdots|v_n)$ . Then P is an orthogonal matrix and  $P^T A P$  is the diagonal matrix with blocks



<sup>&</sup>lt;sup>2</sup>Such a matrix always exists.