Geometrization of the Local Langlands Program

McGill

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Part 1

Day One

CHAPTER 1

Towards Geometric Langlands
Speaker: Jared Weinstein

1. Review of Langlands program

The Langlands program connects two sorts of objects, automorphic forms and Galois represensentations, in many different contexts.

1.1. Class field theory. The subjects begins with class field theory, so let's review that (in the special case of $\mathbb{Q}$).

THEOREM 1.1 (Kronecker-Weber). *Every finite abelian extension $K/\mathbb{Q}$ lies within $\mathbb{Q}(\mu_m)$ for some m.*

Dirichlet characters are identified with 1-dimensional Galois representations via the canonical isomorphism $\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q}) \cong (\mathbb{Z}/m)^\times$. In turn, primitive Dirichlet characters are related to characters of $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times\mathbb{R}_{>0}$.

Stringing these together, we get a correspondence $\chi \leftrightarrow \sigma_\chi$ between

$$\left\{\text{characters of } \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times\mathbb{R}_{>0}\right\} \leftrightarrow \left\{\text{1-dimensional Galois representations}\right\}$$

determined by

$$\chi(1,\dots,p,\dots,1) = \sigma_\chi(\mathrm{Frob}_p).$$

It felt slightly unsatisfying that we had to mod out by $\mathbb{R}_{>0}$. We're going to enhance the picture. Pick a prime ℓ and an identification $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. We now consider *Hecke characters*, i.e. characters on the idele class group $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times$, which are *algebraic* in the sense that $a \mapsto \chi(a)|a|^k$ where χ is a Dirichlet character and $k \in \mathbb{Z}$. This algebraicity condition must be imposed to get a Galois representation.

On the other side, we consider characters $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \overline{\mathbb{Q}}_\ell^\times$ which are *de Rham at ℓ*, i.e. of the form

$$s \mapsto \sigma_\chi(s)\chi_\ell(s)^k.$$

Note that this has infinite order.

1.2. Elliptic modular forms. The next case corresponds to $G = \mathrm{GL}_2$. The "automorphic" objects we consider are normalized cuspidal newforms of weight $k \geq 2$. This means the q-expansion is $f(q) = \sum_{n=1}^\infty a_n q^n$ with $a_1 = 1$. Implicit in "newform" is the requirement that f is a Hecke eigenform, i.e. $T_p f = a_p f$ for "good" p.

Work of Eichler-Shimura and Deligne attaches to such an f an irreducible odd representation $\sigma_f\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$, which is de Rham at ℓ with Hodge-Tate weights $0, k-1$, and determined by the compatibility

$$\mathrm{char}(\sigma_f(\mathrm{Frob}_p)) = T^2 - a_p T + p^{k-1}\epsilon_f(p).$$

To get a better sense of this formula, we want to see how it generalizes. The datum of such an f generalizes to the datum of a cuspidal automorphic representation $\pi \simeq \otimes'_v \pi_v$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, such that π_∞ is discrete series.

This latter is contained in the set of cuspidal *algebraic* automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. On the other side one finds Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ which are de Rham at ℓ (dropping conditions on HT weights and oddness).

$$\begin{array}{ccc}
\left\{\begin{matrix}\text{cuspidal automorphic}\\ \text{representations } \pi \simeq \otimes'_v \pi_v\end{matrix} \mid \pi_\infty \text{ discrete series}\right\} & \longleftrightarrow & \left\{\begin{matrix}\text{Galois representations}\\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)\end{matrix} \mid \begin{matrix}\text{de Rham at } \ell\\ \text{HT weights } 0, k-1\end{matrix}\right\} \\
\big\downarrow & & \big\downarrow \\
\left\{\begin{matrix}\text{algebraic cuspidal}\\ \text{automorphic representations}\end{matrix}\right\} & \longleftrightarrow & \left\{\begin{matrix}\text{Galois representations}\\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)\end{matrix} \mid \text{de Rham at } \ell\right\}
\end{array}$$

In the smaller case (top row), one has some converse statements. In the bigger case, very little is known in either direction.

1.3. Global Langlands for general G. Let G be a split reductive group over $\mathbb{Q}$, e.g. $G = \mathrm{GL}_n, \mathrm{GSp}_{2n}, O_n$. Then there is a Langlands dual group $\widehat{G}$ (obtained by dualizing the root datum).

The "automorphic side" is

$$\left\{\begin{matrix}\text{algebraic cuspidal automorphic}\\ \text{rep'ns of } G(\mathbb{A}_\mathbb{Q})\end{matrix}\right\}$$

Here the adjective "automorphic" means (roughly) "appears in $L^2(G(\mathbb{A}_\mathbb{Q})/G(\mathbb{Q}), \mathbb{C})$". The adjective "algebraic" means that the infinitesimal character of π_∞ agrees with the character of a (finite-dimensional) algebraic representation of $G(\mathbb{R})$.

The conjecture is that there is a map $\pi \mapsto \sigma_\pi$ from the automorphic side to the "Galois side" of Langlands parameters $\sigma\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ which are de Rham at ℓ.

$$\left\{\begin{matrix}\text{algebraic cuspidal automorphic}\\ \text{rep'ns of } G(\mathbb{A}_\mathbb{Q})\end{matrix}\right\} \to \left\{\begin{matrix}\text{Langlands parameters}\\ \sigma\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)\\ \text{de Rham at } \ell\end{matrix}\right\}$$

What is the compatibility in this case? It is some relation between π_p and $\sigma(\mathrm{Frob}_p)$.

For almost all p, π_p is an unramified smooth representation of $G(\mathbb{Q}_p)$. On the other side, $\sigma(\mathrm{Frob}_p)$ should be a semisimple conjugacy class in $\widehat{G}(\overline{\mathbb{Q}}_\ell)$. There is a bijection between

$$\left\{\begin{matrix}\text{unramified smooth}\\ \text{representation of } G(\mathbb{Q}_p)\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{semisimple conjugacy}\\ \text{class in } \widehat{G}(\overline{\mathbb{Q}}_\ell)\end{matrix}\right\},$$

which is given by the Satake equivalence. The compatibility is that π_p matches up with $\sigma(\mathrm{Frob}_p)$ under this bijection. (However, the conjugacy class $\sigma(\mathrm{Frob}_p)$ is not even known to be semi-simple in general.)

1.4. Satake equivalence. Let F be a local nonarchimedean field, e.g. $\mathbb{Q}_p$. Let $\mathcal{O}_F$ be its ring of integers. Let $G/\mathcal{O}_F$ be a split connected reductive group.

Then $G(F)$ is a locally profinite group. Choose a Haar measure μ.

Let $G(\mathcal{O}_F) \subset G(F)$ be a maximal compact open subgroup, e.g. $\mathrm{GL}_n(\mathbb{Z}_p) \subset \mathrm{GL}_n(\mathbb{Q}_p)$.

Definition 1.2. Let (π, V) be a representation of $G(F)$ on a $\mathbb{C}$-vector space. (We view $\mathbb{C}$ as having the discrete topology.) We say π is *smooth* if for all $\in V$, $\mathrm{Stab}(v) \subset G(F)$ is open. (Equivalently, $G(F) \times V \to V$ is continuous, where V has the discrete topology.)

We say π is *admissible* if $\dim V^K < \infty$ for all compact open $K \subset G(F)$.

The Hecke algebra is $\mathcal{H} := C_c^\infty(G(F), \mathbb{C})$ (functions with locally constant, compact supports). A smooth representation π induces an $\mathcal{H}$-module structure on V, via

$$\pi(f)v = \int_{g \in G(F)} f(g)\pi(g)v \, d\mu(g).$$

Since $\mathcal{H} = \bigcup_K C_c^\infty(K\backslash G(F)/K, \mathbb{C})$, we have $f \in C_c^\infty(K\backslash G(F)/K, \mathbb{C})$ for some K, and then $\pi(f)$ acts on V^K.

Definition 1.3. An irreducible smooth admissible representation (π, V) is *unramified* (aka "spherical") if $V^{G(\mathcal{O}_F)} \neq 0$.

We have a bijection

$$\left\{\text{unramified rep'ns of } G(F)\right\} \leftrightarrow \left\{\begin{matrix}\text{irreducible rep'ns of} \\ \mathcal{H}^{\mathrm{unr}} := C_c^\infty(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}), \mathbb{C})\end{matrix}\right\}.$$

Now, $\mathcal{H}^{\mathrm{unr}}$ turns out to be commutative (and unital), and therefore its finite-dimensional irreducible representations are 1-dimensional.

Example 1.4. We will give an explicit presentation of $\mathcal{H}^{\mathrm{unr}}$ for $G = \mathrm{GL}_2/\mathbb{Q}_p$. First, note that

$$\mathrm{GL}_2 = \coprod_{a_1, a_2 \in \mathbb{Z}; a_1 \geq a_2} \mathrm{GL}_2(\mathbb{Z}_p) \begin{pmatrix} p^{a_1} & 0 \\ 0 & p^{a_2} \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p).$$

Let T_{a_1,a_2} be the characteristic function of $\mathrm{GL}_2(\mathbb{Z}_p) \begin{pmatrix} p^{a_1} & 0 \\ 0 & p^{a_2} \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p)$. So $\mathcal{H}^{\mathrm{unr}}$ is generated by T_{a_1,a_2}. What are the relations?

Note $T_{1,1}$ is invertible, with $T_{1,1}^{-1} = T_{-1,-1}$. Also $T_{1,1}T_{a_1,a_2} = T_{a_1+1,a_2+1}$. Therefore $\mathcal{H}^{\mathrm{unr}}$ is generated by operators of the form $T_{1,1}^{\pm}$ and $T_{a,0}$. (In terms of classical modular forms, $T_p \leftrightarrow T_{1,0}$ and $\langle p \rangle \leftrightarrow T_{1,1}$, $T_{p^n} \leftrightarrow T_{n,0}$.) There are also relations.

So unramified representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ are in bijection with characters of $\mathcal{H}^{\mathrm{unr}}$, which is $\mathbb{C}^\times \times \mathbb{C}$. This can be identified with $(\mathbb{C}^\times \times \mathbb{C}^\times)/S_2$, via

$$(\alpha\beta, \alpha+\beta) \leftrightarrow (\alpha, \beta)$$

which in turn is the same as $(\mathrm{GL}_2(\mathbb{C}))^{\mathrm{ss}}/\mathrm{GL}_2(\mathbb{C})$, where the quotient is for the conjugation action.

This (in particular the last few steps) may have seemed random.

THEOREM 1.5 (Satake). *$\mathcal{H}^{\mathrm{unr}} \simeq R(\widehat{G})$, the representation ring of $\widehat{G}$.*

REMARK 1.6. Taking the character of a representation induces an isomorphism between $R(\widehat{G})s$ and $\mathbb{C}[\widehat{G}]^{\widehat{G}}$. One doesn't need $\mathbb{C}$ here – it suffices to work with any algebra over $\mathbb{Z}[q^{\pm 1/2}]$.

2. Geometric Langlands

We carry everything over to function fields, i.e. $F = \mathbb{F}_q(X)$ for a geometrically connected smooth projective curve $X/\mathbb{F}_q$.

2.1. Class field theory. We'll explain how to find the Hilbert class field of F, under the assumption that there is a rational point $\infty \in X(\mathbb{F}_q)$. Unramified extensions of F correspond to unramified covers of X. One source of such comes from $\overline{\mathbb{F}}_q/\mathbb{F}$. The ones orthogonal to this could be described as being totally split over ∞, and we denote the maximal such by $Y \to X$. So the maximal unramified cover of X is the compositum of Y and $X_{\overline{\mathbb{F}}_q}$.

Consider the embedding $X \to \mathrm{Jac}(X) = \mathrm{Pic}^0(X)$ given by $P \mapsto P - (\infty)$. There is a Lang isogeny $L\colon \mathrm{Jac}(X) \to \mathrm{Jac}(X)$ given by $L(x) = \mathrm{Frob}_q(x) - x$. So $\ker(L) = \mathrm{Jac}(\mathbb{F}_q) = \mathrm{Pic}^0(X)$. It turns out that we have a Cartesian square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \mathrm{Jac}(X) \arrow[d, "L"] \\ X \arrow[r] & \mathrm{Jac}(X) \end{tikzcd}$$

We can present

$$\mathrm{Pic}(X) \simeq \mathbb{A}_F^\times/(F^\times \prod_{v\in|X|} \mathcal{O}_{F_v}^\times)$$

2.2. The case of GL_n.

THEOREM 2.1 (Drinfeld, Lafforgue). *There is a bijection between*

$$\left\{ \begin{matrix} \textit{cuspidal automorphic} \\ \textit{representations of } \mathrm{GL}_n(\mathbb{A}_F) \\ \textit{with finite order central character} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \textit{irreducible representations} \\ \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \\ \textit{with finite order determinant} \end{matrix} \right\}.$$

Important fact: $\mathrm{Pic}(X)$ is the $\mathbb{F}_q$-points of a groups cheme $\mathrm{Pic}\, X$. What is the higher rank version of this? We want to realize the double coset space

$$\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/\prod_{v\in|X|}\mathrm{GL}_n(\mathcal{O}_{F_v})$$

as rational points of something. The observation is that this is identified with the set of vector bundles of rank n on X. (For us, a vector bundle is a locally free $\mathcal{O}_X$-module.) Why?

Here is a sketch of the direction $\leftarrow$. Given a vector bundle $\mathcal{E}$, choose trivializations $\alpha\colon \mathcal{E}|_U \xrightarrow{\sim} \mathcal{O}_U^n$ for some dense open $U \subset X$. Also $\mathcal{E}|_{\mathrm{Spec}\,\mathcal{O}_v}$ is free, and we can choose a trivialization $\beta\colon \mathcal{E}|_{\mathrm{Spec}\,\mathcal{O}_v} \xrightarrow{\sim} \mathcal{O}_v^n$. Then $\alpha_v \circ \beta_v^{-1} =: \gamma_v \in \mathrm{GL}_n(F_v)$. The (γ_v) determine an element of $\mathrm{GL}_n(\mathbb{A}_F)$.

Definition 2.2. Let Bun_n be the stack taking an $\mathbb{F}_q$-scheme S to the groupoid of rank n vector bundles on $X \times_{\mathbb{F}_q} S$.

Theorem 2.3. Bun_n *is a smooth Artin stack (of dimension* $(g-1)n^2$*?).*

Example 2.4. $\mathrm{Bun}_1(S)$ classifies lie bundles on $X \times S$. Any line bundle has automorphisms $\mathbb{G}_m$.

An Artin stack $\mathcal{X}$ is a stack that has a smooth uniformization $U \to \mathcal{X}$. The "smooth" adjective means that we can take U to be smooth as well.

We always have $\mathrm{Bun}_n = \coprod_{d\in\mathbb{Z}} \mathrm{Bun}_n^d$. In particular, there are infinitely many connected components if G is not semisimple. Even individual connected components may fail to be quasi-compact.

Example 2.5. For $n=2$, $X=\mathbb{P}^1$, let Bun_2^0 be the degree 0 component. Any $\mathcal{E}$ on $X \times S$ looks like $\mathcal{O}(n)\oplus\mathcal{O}(-n)$ pointwise on S.

Define $\mathrm{Bun}_2^{0,\leq h}$ to classify bundles which are pointwise on S of the form $\mathcal{E} \simeq \mathcal{O}(k)\oplus\mathcal{O}(-k)$ for $0 \leq k \leq h$. So we get an open substack $\mathrm{Bun}_2^{0,\leq h} \subset \mathrm{Bun}_2^0$. We could rephrase this condition as "$\mathcal{E}(h)$ is generated by global sections, and $\dim H^0 = 2h+2$."

We can make a scheme U that (roughly speaking) parametrizes $\{\mathcal{O}_X^{2h+2} \twoheadrightarrow \mathcal{F}\}$. This is a quote scheme, and we can use a (smooth open subset of) it to uniformize $\mathrm{Bun}_2^{0,\leq h}$.

The geometrization shifts focus from the space of automorphic functions, $C(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/\prod_v \mathrm{GL}_n(\mathcal{O}_{F_v}))$, to the category of ℓ-adic sheaves on Bun_n.

Theorem 2.6 (Frenkel-Gaitsgory-Vilonen). *There is a bijection between*

$$\left\{\begin{matrix}\textit{cuspidal eigensheaves}\\ \textit{on } \mathrm{Bun}_n\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\textit{geometrically irreducible}\\ n\textit{-dimensional}\\ \ell\textit{-adic local systems on } X\end{matrix}\right\}.$$

CHAPTER 2

Lightning introduction to p-adic geometry
Speaker: David Hansen

1. Adic spaces

1.1. Huber rings.

DEFINITION 1.1. A *Huber ring* is a (Hausdorff) topological ring A containing an open subring A_0 such that the topology on A_0 coincides with the I-adic topology for some finitely generated ideal $I \subset A_0$. (There could be more than one choice for A_0.)

We say A_0 is a "ring of definition" and (A, I) is "a couple of definition".

DEFINITION 1.2. We let $A^\circ \subset A$ be the subring of power-bounded elements.

EXAMPLE 1.3. For $A = \mathbb{Q}_p$ with the p-adic topology, $A^\circ = \mathbb{Z}_p$. We can take $A_0 = \mathbb{Z}_p$ and $I = (p)$.

DEFINITION 1.4. Given A, a *ring of integral elements* is an open and integrally closed subring $A^+ \subset A$ with $A^+ \subseteq A^0$.

DEFINITION 1.5. A *Huber pair* is a pair (A, A^+).

DEFINITION 1.6. We say that a Huber ring A is a *Tate ring* if it contains a topologically nilpotent unit.

1.2. Valuations.

DEFINITION 1.7. Given a topological ring A, a *continuous valuation* on A is a function $|\cdot|\colon A \to \Gamma \cup \{0\}$ where Γ is a totally ordered abelian group, satisfying:

(1) $|ab| = |a| \cdot |b|$ and $|a + b| \leq \max(|a|, |b|)$,
(2) $|0| = 0$ and $|1| = 1$,
(3) for all γ in the image of $|\cdot|$, the subset $\{a \in A\colon |a| < \gamma\}$ is open in A.

We say $|\cdot|$ and $|\cdot|'$ are equivalent if $|a| \leq |b| \iff |a|' \leq |b|'$ for all $a, b \in A$.

1.3. Adic spectrum.

DEFINITION 1.8. Given (A, A^+) define the *adic spectrum* $\mathrm{Spa}(A, A^+)$ to be the set of equivalence classes of continuous valuations $|\cdot|$ on A such that $|a| \leq 1$ for all $a \in A^+$. For $x \in \mathrm{Spa}(A, A^+)$ write $|\cdot|_x\colon A \to \Gamma_x \cup \{0\}$ for a choice of valuation representing the equivalence class. Give this the topology whose open subsets are generated by

$$\{x \in \mathrm{Spa}(A, A^+)\colon |f|_x \leq |g|_x \neq 0\}, \quad f, g \in A.$$

THEOREM 1.9. $\mathrm{Spa}(A, A^+)$ *is a spectral space. (There are several equivalent definitions, e.g. it coincides with* $\mathrm{Spec}\, R$ *equipped with the Zariski topology, for some* R*.)*

REMARK 1.10. In particular, $\mathrm{Spa}(A, A^+)$ is always quasicompact with a basis of quasicompact opens.

1.4. Rational subsets.

DEFINITION 1.11. Let $X = \mathrm{Spa}(A, A^+)$ and $s \in A$ be arbitrary. Let $T \subset A$ be any finite subset generating an open ideal in A. A *rational subset* of X is one of the form

$$U\left(\frac{T}{s}\right) = \{x \in X \colon |t|_x \leq |s|_x \neq 0 \text{ for all } t \in T\}.$$

Rational subsets are open, quasi-compact, and stable under finite intersection, and they generate the topology on X.

PROPOSITION 1.12. *If* $U \subset X = \mathrm{Spa}(A, A^+)$ *is a rational subset, then there exists a complete Huber pair* (A_U, A_U^+) *with a map* $\varphi \colon (A, A^+) \to (A_U, A_U^+)$ *such that* $\mathrm{Spa}(A_U, A_U^+) \to X$ *is a homeomorphism onto* U*, and such that* φ *is universal for maps from* (A, A^+) *to complete Huber pairs which factor over* U *on adic spectra.*

EXAMPLE 1.13. $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ has two points: a "generic point" η corresponding to the p-adic valuation, and a "special point" s which factors through the trivial valuation on $\mathbb{F}_p$.

EXAMPLE 1.14. Let $(A, A^+) := (\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$. The adic spectrum is "the closed unit disk over $\mathbb{Q}_p$". Then

$$U\left(\frac{\{T, p\}}{p}\right) = \{|T|_x \leq |p|_x \neq 0\}$$

is "the subdisk of radius $1/p$".

The universal property implies that (A_U, A_U^+) is unique up to unique isomorphism. It also implies that whenever $U \subset V$ is an inclusion of rational subsets, one gets $(A_V, A_V^+) \to (A_U, A_U^+)$.

DEFINITION 1.15. Given $X = \mathrm{Spa}(A, A^+)$ we defines the *structure presheaf* $\mathcal{O}_X$ by

$$\mathcal{O}_X(U) = \varprojlim_{W \text{ rational } \subset U} A_W.$$

We also the *integral structure presheaf* $\mathcal{O}_X^+$ similarly:

$$\mathcal{O}_X^+(U) = \varprojlim_{W \text{ rational } \subset U} A_W^+.$$

These are presheaves of complete topological rings. For all $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a *local* ring, and the valuation $|\cdot|_x$ extends to a valuation $\mathcal{O}_{X,x} \to \Gamma_x \cup \{0\}$ (whose kernel is the maximal ideal).

WARNING 1.16. $\mathcal{O}_X$ is not always a sheaf. (When it is a sheaf, then $\mathcal{O}_X^+$ is also a sheaf.)

PROPOSITION 1.17. *In each of the following situations, the structure presheaf on* $\mathrm{Spa}(A, A^+)$ *is a sheaf. (In (3)-(5), assume that A is Tate.)*

(1) A is discrete. [This encompasses schemes]
(2) A admits a Noetherian ring of definition. [This encompasses formal schemes]
(3) A is strongly Noetherian*, i.e.* $A\langle X_1, \ldots, X_n\rangle$ *is Noetherian for any n. [This encompasses rigid analytic varieties]*
(4) A is stably uniform*, i.e. for all rational subset* $U \subset \mathrm{Spa}(A, A^+)$ *the subring* $A_U^\circ \subset A_U$ *is bounded.*
(5) A is perfectoid*: i.e. A is complete,* A° *is a bounded subring of A, there exists a topologically nilpotent unit* $\varpi \in A$ *such that* $\varpi^p \mid p$*, and the Frobenius map*

$$\Phi\colon A^\circ/\varpi \to A^\circ/\varpi^p$$

is surjective. (This last continuous is equivalent to the Frobenius on A°/p *being surjective.)*

REMARK 1.18. The fact that $\mathcal{O}_X$ is a sheaf in (5) was initially proved directly by Scholze. Later condition (4) was discovered Buzzard-Verberkmoes and (5) is a special case of it.

EXAMPLE 1.19. The following are examples of perfectoid rings: $\mathbb{C}_p, \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}, \mathbb{C}_p\langle T^{1/p^\infty}\rangle$.

An adic space is a space glued locally from the adic spectrra of sheafy Huber pairs.

2. Perfectoid spaces

DEFINITION 2.1. A *perfectoid space* is a space glued locally from the adic spectra of perfectoid Huber pairs.

2.1. Tilting.

DEFINITION 2.2. Let A be a complete topological ring in which p is topologically nilpotent. The *tilt* of A is

$$A^\flat := \varprojlim_{x \mapsto x^p} A = \{x = (x_0, x_1, x_2, \ldots) \in A^{\mathbb{N}} \mid x_{i+1}^p = x_i \text{ for all } i \geq 0\}$$

with the inverse limit topology. Make this into a topological ring via coordinate-wise multiplication, and addition law

$$(x+y)_i := \lim_{n\to\infty} (x_{i+n} + y_{i+n})^{p^n}.$$

EXAMPLE 2.3. For rings which are not "big enough", i.e. don't have many pth power roots, $A^\flat$ is too small to be interesting, e.g. $\mathbb{Q}_p^\flat = \mathbb{F}_p$.

2.2. Tilting equivalence. Note: there is a canonical map $\#\colon A^\flat \to A$ sending $x \mapsto x_0$.

PROPOSITION 2.4. *If (A, A^+) is a perfectoid Huber pair, then $(A^\flat, A^{+\flat})$ is a perfectoid Huber pair in characteristic p. Moreover, there is a canonical homeomorphism $\mathrm{Spa}(A, A^+) \simeq \mathrm{Spa}(A^\flat, A^{+\flat})$ taking $|\cdot|_x \mapsto |\cdot|_x \circ \#$, which*

(1) identifies rational subsets $U \leftrightarrow U^\flat$,
(2) induces $\mathcal{O}_X(U)^\flat \cong \mathcal{O}_{X^\flat}(U^\flat)$.

In particular, this operation glues to a functor $X \mapsto X^\flat$ from perfectoid spaces to perfectoid spaces in characteristic p.

PROPOSITION 2.5. *Given a perfectoid space X, tilting induces an equivalence of categories*

$$\{\text{perfectoid spaces } Y/X\} \leftrightarrow \{\text{perfectoid spaces } Y^\flat/X^\flat\}.$$

EXAMPLE 2.6. If $A = \mathbb{C}_p$, then $A^\flat \simeq \widehat{\overline{\mathbb{F}_p((t))}}$. If $A = K\langle T^{1/p^\infty}\rangle$ for K a perfectoid field, then $A^\flat \simeq K^\flat\langle T^{1/p^\infty}\rangle$.

3. Diamonds

3.1. Étale morphisms.

DEFINITION 3.1. A map $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ is *finite étale* if $A \to B$ is finite étale and B^+ is the integral closure of A^+ in B.

In general, a map of adic spaces $f\colon X \to Y$ is *finite étale* if there exists an open covering $Y = \bigcup Y_i$ such that $f^{-1}(Y_i)$ is affinoid and $f^{-1}(Y_i) \to Y_i$ is finite étale for all i.

DEFINITION 3.2. A map of adic spaces $f\colon X \to Y$ is étale if locally on some open covering $X = \bigcup X_i$, f can be factored as

$$\begin{tikzcd} X_i \arrow[r, hook, "j_i"] & V_i \arrow[d, "g_i"] & \\ & W_i \arrow[r, hook, "h_i"] & Y \end{tikzcd}$$

where j_i and h_i are open embeddings, and g_i is finite étale. (Note that the analogous statement is false for schemes.)

FACT 3.3. *If $X \to Y$ is étale and Y is perfectoid then X is perfectoid too. (This is related to the almost purity theorem.)*

REMARK 3.4. It is not known that a perfectoid affinoid space, i.e. an affinoid space which is perfectoid, is necessary the adic spectrum of a perfectoid ring.

3.2. Pro-étale site.

DEFINITION 3.5. A map $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ of affinoid perfectoid spaces is *affinoid pro-étale* if

$$(B, B^+) = \widehat{\varinjlim_i (A_i, A_i^+)}$$

where (A_i, A_i^+) is perfectoid and $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$ is étale.

A map of perfectoid spaces $f\colon X \to Y$ is *pro-étale* if locally on affinoid covers of X and Y, it is affinoid pro-étale.

DEFINITION 3.6. A pro-étale map of perfectoid spaces $f\colon X \to Y$ is a *pro-étale cover* if for all quasicompact opens $U \subset Y$ there exists a quasicompact open $V \subset X$ such that $f(V) = U$.

REMARK 3.7. Perfectoid spaces with the Grothendieck topology of pro-étale covers form a site.

DEFINITION 3.8. The *big pro-étale site* is the site Perf with objects characteristic p perfectoid spaces, and covers being pro-étale covers.

PROPOSITION 3.9. *This site is sub-canonical, i.e. representable presheaves* $h_X = \mathrm{Hom}(-, X)$ *are sheaves.*

3.3. Diamonds.

DEFINITION 3.10. A *diamond* is a sheaf D on Perf such that

$$D \cong \mathrm{Coeq}(R \rightrightarrows X),$$

where $R \rightrightarrows X$ is an equivalence relation in characteristic p perfectoid spaces, where the maps are pro-étale.

We let Dia be the category of such objects.

3.3.1. *Underlying topological space.* Any diamond D has a canoncially associated topological space $|D|$. If $D \simeq \mathrm{Coeq}(R \rightrightarrows X)$, then $|R| \rightrightarrows |X|$ is an equivalence relation and $|D| = |X|/|R|$ with the quotient topology. (One could say that $|\cdot|\colon \mathrm{Dia} \to \mathrm{Top}$ is the left Kan extension of $|\cdot|\colon \mathrm{Perf} \to \mathrm{Top}$ along $\mathrm{Perf} \hookrightarrow \mathrm{Dia}$.)

PROPOSITION 3.11. *There is a canonical bijection between*

$$\{\textit{open subsets of } |D|\} \leftrightarrow \{\textit{open subdiamonds of } D\}$$

3.3.2. *Analytic adic spaces.* There is a natural functor from (most) adic spaces over $\mathbb{Z}_p$ to Dia.

DEFINITION 3.12. Let X be an adic space. We say X is *analytic* if for all $x \in X$, the topology induced by $|\cdot|_x$ on $\mathcal{O}_{X,x}/\mathfrak{m}_x$ to be non-discrete.

PROPOSITION 3.13. *There is a canonical functor*

$$\left\{\begin{matrix} \textit{analytic adic spaces} \\ \textit{over } \mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p) \end{matrix}\right\} \to \mathrm{Dia}$$

denoted $X \mapsto X^\diamond$.

(1) *This extends the functor $X \mapsto h_{X^\flat}$ on perfectoid spaces.*
(2) *There is a functorial homeomorphism $|X| \cong |X^\diamond|$.*
(3) *Moreover, $(-)^\diamond$ induces an equivalence $X_{\acute{e}t} \cong X^\diamond_{\acute{e}t}$ for appropriate definition of these sites.*

To define $X^\diamond\colon \mathrm{Perf} \to \mathrm{Sets}$, set

$$T \mapsto \left\{ (T^\#, \iota, f)\colon \begin{array}{r} T^\# = \text{ a perfectoid space} \\ \iota\colon T^{\#\flat} \xrightarrow{\sim} T \\ f\colon T^\# \to X \end{array} \right\}.$$

Informally, $X^\diamond$ takes T to the "set of untilts of T".

What is not obvious is that this is a diamond.

Lemma 3.14 (Faltings, Colmez). *Let (A, A^+) be a complete Huber pair over $\mathbb{Z}_p$. Then there exists a filtered directed system (A_i, A_i^+) of finite étale (A, A^+)-algebras, Galois with Galois group G_i, such that the completed direct limit $\varinjlim (A_i, A_i^+) =: (A_\infty, A_\infty^+)$ is perfectoid, and the map $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$ is a $\underline{G}$-torsor where $G := \varprojlim G_i$.*

Morally, $\mathrm{Spa}(A_\infty, A_\infty^+)$ is a perfectoid G_∞-cover of $\mathrm{Spa}(A, A^+)$. What you actually have to prove is that $\mathrm{Spa}(A, A^+)^\diamond \cong \mathrm{Spa}(A_\infty^\flat, A_\infty^{+\flat})/\underline{G}$, and the latter is a diamond.

Example 3.15. A presentation of $\mathrm{Spa}\, \mathbb{Q}_p^\diamond$ is

$$\mathrm{Spa}\, \mathbb{Q}_p^\diamond = \mathrm{Spa}(\widehat{\mathbb{Q}_p(\zeta_{p^\infty})})^\flat / \mathbb{Z}_p^\times.$$

3.4. Slogans.

Claim: Any reasonable "topological" property of schemes has an analogue for diamonds.

In particular, one can make sense of diamonds as being quasicompact, quasiseparated, separated, etc. and maps of diamonds as being quasicompact, quasiseparated, separated, open, immersions, proper, etc.

The only twist: a diamond can be simultaneously separated and non-quasi-separated, because the notion of separatedness refers to a point, which is not a diamond. The main difference between schemes and diamonds is that non-quasi-separated stuff appears naturally.

3.5. Spatial diamonds.

Definition 3.16. A diamond D is *spatial* if it is qcqs and the subsets $|U| \subset |D|$, where $U \subset D$ varies over quasicompact open subdiamonds, gives a neighborhood basis.

We say D is *locally spatial* if it has an open cover by spatial subdiamonds.

Proposition 3.17.

(1) *If D is a (locally) spatial diamond then $|D|$ is a (locally) spectral space.*
If $D \to E$ is a map of locally spatial diamonds, then $|D| \to |E|$ is nice (i.e. spectral and generalizing).

(2) If $X \to Y \leftarrow Z$ is a diagram of (locally) spatial diamonds, then $X \times_Y Z$ is (locally) spatial.

(3) Any $X^\diamond$ is locally spatial, and is spatial if and only if X is qcqs.

(4) If X is locally spatial, then $X_{ét}$ is well-behaved.

3.6. Principles of creation. If you have some diamonds, how do you make more diamonds?

(1) Fiber products and direct products of diamonds are diamonds. (These are different since diamonds don't have a final object.)

(2) Anything pro-étale over or under a diamond is a diamond. In particular, (coarse) quotients of diamonds by actions of locally profinite groups are diamonds.

(3) Any reasonable subsets of $|D|$, for D a diamond, give rise to reasonable subdiamonds of D.

Part 2

Day Two

CHAPTER 3

The Fargues-Fontaine Curve
Speaker: Jared Weinstein

1. Analytic adic spaces

Recall that a Huber ring A is *Tate* if it contains a topologically nilpotent unit ϖ (which will be referred to as a "pseudo-uniformizer").

DEFINITION 1.1. A point x in an adic space is *analytic* if there exists a rational neighborhood $U = \mathrm{Spa}(A, A^+)$ of x where A is Tate.

Let A be a complete Tate ring, ϖ a pseudo-uniformizer of A, A_0 a ring of definition. Then we can define a norm

$$|\cdot|\colon A \to \mathbb{R}_{>0}$$

by

$$|a| = \inf_{n\in\mathbb{Z}\colon \varpi^n a \in A_0} 2^n.$$

This induces the topology on A. Therefore, Tate rings are Banach.

1.1. The rank one generization. If $x \in \mathrm{Spa}(A, A^+)$ corresponds to $|\cdot|_x\colon A \to \Gamma$, then $\gamma = |\varpi|_x = |\varpi(x)| \in \Gamma$ must satisfy $\gamma^n \to 0$ as $n \to \infty$. There exists a map $\Gamma \to \mathbb{R}_{>0}$ sending $\gamma \mapsto \frac{1}{2}$. Then we define a new valuation

$$|\cdot|_{\widetilde{x}}\colon A \xrightarrow{|\cdot|_x} \Gamma \to \mathbb{R}_{>0}.$$

The corresponding $\widetilde{x} \in \mathrm{Spa}(A, A^+)$ is an $\mathbb{R}_{>0}$-valued (rank 1) point which specializes to x, i.e. $\widetilde{x} \rightsquigarrow x$. Then $|\cdot|_{\widetilde{x}} \leq |\cdot|$; the set of rank 1 points of $\mathrm{Spa}(A, A^+)$ coincides with the set of rank 1 valuations $\leq |\cdot|$.

The point $\widetilde{x}$ doesn't depend on the choice of ϖ. If ϖ' is another pseudo-uniformizer, then

$$\frac{\log|\varpi(\widetilde{x})|}{\log|\varpi'(\widetilde{x})|} \in \mathbb{R}_{>0}.$$

Notation: if A is a Huber ring, we may abbreviate $\mathrm{Spa}\, A = \mathrm{Spa}(A, A^\circ)$.

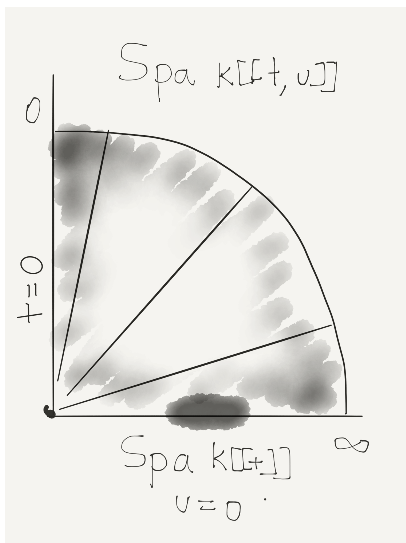
EXAMPLE 1.2. Let $k \subset \overline{\mathbb{F}}_p$. Then $\mathrm{Spa}\, k = \{s\}$, which is not analytic.

EXAMPLE 1.3. $\mathrm{Spa}\, k[[t]] = \{s, \eta\}$ then the analytic locus is η. (Any adic space has a maximal analytic locus, which is an open subspace.)

EXAMPLE 1.4. Consider $\mathrm{Spa}(k[[t]]) \times_{\mathrm{Spa}\, k} \mathrm{Spa}(k[[u]]) = \mathrm{Spa}\, k[[t,u]]$. This contains a special point s such that $|t(s)| = |u(s)| = 0$. Outside s, at least one of u or t is non-vanishing, hence a unit, and both are topologically nilpotent. So s is the only non-analytic point. The complement $\mathcal{Y}$ is covered by two rational subsets

$$U(|t| \le |u| \neq 0) = \mathrm{Spa}(k((u))\langle \frac{t}{u} \rangle, k[[u]]\langle \frac{t}{u} \rangle),$$
$$U(|u| \le |t| \neq 0) = \mathrm{Spa}(k((t))\langle \frac{u}{t} \rangle, k[[t]]\langle \frac{u}{t} \rangle),$$

Given $x \in \mathcal{Y}$, let $\kappa(x) = \frac{\log|u(\widetilde{x})|}{\log|t(\widetilde{x})|} \in [0,\infty]$. This defines a continuous, surjective map $\kappa \colon \mathcal{Y} \to [0,\infty]$.



What is $\mathrm{Spa}\, k((t)) \times_{\mathrm{Spa}\, k} \mathrm{Spa}((u))$? It is the open subset where $tu \neq 0$, i.e. $\kappa^{-1}(0,\infty)$. This is not quasi-compact. In particular this shows $\mathrm{Spa}\, k$ is not quasi-separated.

EXAMPLE 1.5. The space $\mathrm{Spa}(\mathbb{Z}_p[[t]], \mathbb{Z}_p[[t]])$ is similar. Morally, $\mathbb{Z}_p[[t]] =$ "$\mathbb{F}_p[[p,t]]$".

2. The Fargues-Fontaine curve

2.1. The adic Fargues-Fontaine curve. Let $C/\mathbb{F}_p$ be an algebraically closed perfectoid field, e.g. $C = \mathbb{C}_p^\flat$.

DEFINITION 2.1. Let $A_{\mathrm{inf}} = W(\mathcal{O}_C)$, with its $(p,[\varpi])$-adic topology, $0 < |\varpi| < 1$. Morally, "$A_{\mathrm{inf}} = \mathcal{O}_C[[p]]$".

Let $\mathcal{Y} = \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) \setminus \{s\}$, where s is the point such that $|\varpi(s)| = |p(s)| = 0$. Then $\mathcal{Y}$ is an analytic, and there exists $\kappa \colon \mathcal{Y} \to [0,\infty]$ as before. The Frobenius ϕ_C acts on $\mathcal{Y}$, and $\kappa(\phi(y)) = p\kappa(y)$. In particular, ϕ acts discontinuously on $\mathcal{Y}_{(0,\infty)} = \kappa^{-1}(0,\infty)$.

Let $I \subset (0,\infty)$ be a closed interval, $B_I = H^0(\kappa^{-1}(I)^\circ, \mathcal{O}_{\mathcal{Y}})$, where $\kappa^{-1}(I)^\circ$ is the interior of $\kappa^{-1}(I)$. A Theorem of Kedlaya says that B_I is strongly noetherian, then $\mathcal{Y}_{(0,\infty)}$ is an adic space.

It turns out that B_I is in fact a PID. Let $B = B_{(0,\infty)} = \varprojlim B_I$. This is a Frechet algebra.

Definition 2.2. The *adic* Fargues-Fontaine curve is $\mathcal{X}_{(C)} := \mathcal{Y}_{(0,\infty)}/\phi_C$.

2.2. Connection to untilts.

Definition 2.3. An *untilt* of C to $\mathbb{Q}_p$ is a pair $(C^\#, \iota)$ where $C^\#/\mathbb{Q}_p$ is a perfectoid field, and $\iota\colon C \xrightarrow{\sim} C^{\#\flat}$, with the obvious notion of equivalence.

Also note that ϕ_C acts on the set of untilts of C to $\mathbb{Q}_p$.

Theorem 2.4. *There is a bijection between untilts of C to $\mathbb{Q}_p$, modulo equivalence, to closed maximal ideals of B.*

Given a maximal ideal $\mathfrak{m}$, one makes the untilt $B/\mathfrak{m}$. Conversely, if $(C^\#, \iota)$ is an untilt, then we have a map of multiplicative monoids (but not a ring homomorphism)

$$\mathcal{O}_C \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_{C^\#} \to \mathcal{O}_{C^\#}.$$

denoted $x \mapsto x^\#$. This lifts to a ring homomorphism

$$W(\mathcal{O}_C) = A_{\text{inf}} \to \mathcal{O}_{C^\#}$$

sending $[f] \mapsto f^\#$. This extends to

$$W(\mathcal{O}_C)[1/p] \to C^\#.$$

Recall that there was a map $W(\mathcal{O}_C)[1/p] \to B_I$. For sufficiently large I, this will extend through B_I, and composing with $B \to B_I$ gives a homomorphism $B \to C^\#$.

3. Untilts

If $(C^\#, \iota)$ is an untilt, then there exists

$$\mathbb{Q}_p(\mu_{p^\infty})^\wedge \hookrightarrow C^\#.$$

Any two such embeddings are related by an automorphism of $\mathbb{Q}_p(\mu_{p^\infty})^\wedge$, and the group of such is $\mathbb{Z}_p^\times$.

Tilting such an inclusion gives

$$\mathbb{Q}_p(\mu_{p^\infty})^{\wedge\flat} \to C$$

which is the same as $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in C$, well defined up to $\epsilon \mapsto \epsilon^a$ for $a \in \mathbb{Z}_p^\times$. We have $|\epsilon - 1| < 1$, so

$$t := \log[\epsilon] = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{([\epsilon]-1)^n}{n} \in B.$$

Now, $\phi([\epsilon]) = [\epsilon^p] = [\epsilon]^p$, so $t \in B^{\phi = p}$, and is well-defined up to multiplication by $\mathbb{Z}_p^\times$.

THEOREM 3.1. *The map* $\epsilon \mapsto \log([\epsilon])$ *induces an isomorphism of* $\mathbb{Z}_p$*-modules:*

$$1 + \mathfrak{m}_{\mathcal{O}_C} \xrightarrow{\sim} B^{\phi=p}.$$

REMARK 3.2. This can even be viewed as an isomorphism of $\mathbb{Q}_p$-vector spaces.

THEOREM 3.3. *The map*

$$\{\mathit{untilts}\} \to (B^{\phi=p} \setminus \{0\})/\mathbb{Z}_p^\times$$

is a bijection, thus

$$\{\mathit{untilts}\}/\phi_C \xrightarrow{\sim} (B^{\phi=p} \setminus \{0\})/\mathbb{Q}_p^\times.$$

4. The scheme-theoretic curve

Recall that we defined the adic Fargues-Fontaine curve

$$\mathcal{X} = \mathcal{Y}_{(0,\infty)}/\phi_c^{\mathbb{Z}}$$

and $B = H^0(\mathcal{Y}_{(0,\infty)}, \mathcal{O}_{\mathcal{Y}_{(0,\infty)}})$ which has an action of ϕ. We make a line bundle $\mathcal{O}(1)$ on $\mathcal{X}$ by descending $\mathcal{O}_{\mathcal{Y}_{(0,\infty)}} e$ where $\phi_C e = p^{-1}e$. Then $H^0(\mathcal{X}, \mathcal{O}(1)) = B^{\phi=p}$.

Let $P := \bigoplus_{n=0}^\infty B^{\phi=p^n}$. (We could interpret $B^{\phi=p^n} = H^0(\mathcal{X}, \mathcal{O}(n))$.)

DEFINITION 4.1. The *schematic Fargues-Fontaine curve* is $X := \operatorname{Proj} P$.

WARNING 4.2. We have $X \to \operatorname{Spec} \mathbb{Q}_p$, but this morphism is far from being of finite type.

THEOREM 4.3.

(1) X is the union of spectra of Dedekind rings. In fact, for all $x \in |X|$ the ring $H^0(X - \{x\}, \mathcal{O}_X)$ is a PID.
(2) (X is "complete") Given $f \in \mathbb{Q}_p(X)$, we get $\operatorname{Div}(f) \in \operatorname{Div}|X|$. *Then* $\deg(\operatorname{Div} f) = 0$.
(3) (Points classify untilts) There is a bijection

$$\{\mathit{untilts\ of\ } C \mathit{\ over\ } \mathbb{Q}_p\}/\phi_C^{\mathbb{Z}} \simeq |X|$$

by taking $x \in |X|$ to its residue field.

REMARK 4.4. Kedlaya and Temkin have showed that for $C = \mathbb{C}_p^\flat$, there exist untilts over $\mathbb{Q}_p$ which are not isomorphic.

5. Vector bundles on X

We have a map $\mathcal{Y}_{(0,\infty)} \to \mathcal{X}$ given by quotienting out by ϕ. We can construct vector bundles on $\mathcal{X}$ by descent.

We will describe a functor

$$\{\text{isocrystals}/k\} \to \{\text{vector bundles}/X\}.$$

We have an embedding $k = \overline{\mathbb{F}}_p \hookrightarrow C$. This induces $W(k) \subset W(\mathcal{O}_C) = A_{\mathrm{inf}}$.

DEFINITION 5.1. An *isocrystal* over k is a vector space V over $W(k)[1/p] =: K$, together with a map $\phi_V \colon V \xrightarrow{\sim} V$ which is ϕ_K-semilinear, i.e. induces an isomorphism $\phi_K^* V \xrightarrow{\sim} V$.

The *Dieudonné-Manin classification* says that the category of isocrystals is semisimple, and the simple ones are of the form K^n with

$$\phi = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ p^d & 0 & \dots & 0 \end{pmatrix} \phi_K$$

where $n \geq 1$ and $d \in \mathbb{Z}$, with $\gcd(d, n) = 1$.

FACT 5.2. *The set of p-divisible groups over k, up to isogeny, is in bijection with the set of isocrystals over k with slopes in* $[0, 1]$.

Given V, we can construct a vector bundle $\mathcal{E}_V$ on $\mathcal{X}$ by descending $V \otimes \mathcal{O}_{Y_{(0,\infty)}}$ along $(\phi_V \otimes \phi_C)$.

EXAMPLE 5.3. The isocrystal of slope -1 is sent to $\mathcal{E}_V = \mathcal{O}(1)$, whose global sections are $B^{\phi=p}$.

5.1. Schematic version. The vector bundle attached to V on the scheme-theoretic incarnation of the Fargues-Fontaine curve corresponds to the graded module

$$\bigoplus_{n=0}^{\infty} (V \otimes_K B)^{\phi=p^n}.$$

6. Link with p-divisible groups

We can define $\mathcal{O}(d/n) = \mathcal{E}_V$ where V is simple of slope $-d/n$.

THEOREM 6.1. *Every vector bundle on X is isomorphic to some $\mathcal{E}_V$.*

Suppose that G/k is the p-divisible group of slope d/n. (For example, if $d/n = 1$ then $G = \mu_{p^\infty}$, while if $d/n = 0$ then $G = \mathbb{Q}_p/\mathbb{Z}_p$.)

We can define

$$G(\mathcal{O}_C) = \varprojlim_n G(\mathcal{O}_C/\varpi^n).$$

EXAMPLE 6.2. If $G = \mathbb{Q}_p/\mathbb{Z}_p$ then $G(\mathcal{O}_C) = \mathbb{Q}_p/\mathbb{Z}_p$, while if $G = \mu_{p^\infty}$ then $G(\mathcal{O}_C) = 1 + \mathfrak{m}_{\mathcal{O}_C}$ (the formal multiplicative group).

THEOREM 6.3. *Assume $d > 0$. Then there is an isomorphism*

$$G(\mathcal{O}_C) \xrightarrow{\sim} H^0(X, \mathcal{O}(d/n)).$$

REMARK 6.4. To get this formula to hold in general (i.e. for all d), you take the universal cover $\widetilde{G} = \varprojlim_p G$, e.g. $\widetilde{\mathbb{Q}_p/\mathbb{Z}_p} = \mathbb{Q}_p = H^0(X, \mathcal{O})$.

7. The diamond formula

Recall that for $S \in \mathrm{Perf}$, then

$$(\mathrm{Spa}\,\mathbb{Q}_p)^\diamond(S) = \left\{ (S^\#, \iota) \mid \begin{array}{c} S^\# = \text{perfectoid space}/\mathbb{Q}_p \\ \iota\colon S \xrightarrow{\sim} S^{\#\flat} \end{array} \right\}$$

We had also defined a functor $Z \mapsto Z^\diamond$ from analytic adic spaces over $\mathbb{Z}_p$ to Dia.

Theorem 7.1. *We have*

$$\mathcal{X}_C^\diamond \simeq (\mathrm{Spa}\, C \times_{\mathrm{Spa}\,\mathbb{F}_p} (\mathrm{Spa}\,\mathbb{Q}_p)^\diamond)/(\phi_C \times 1).$$

That is, if $S \in \mathrm{Perf}$ then the LHS classifies

$$\{S^\#, S^\# \to \mathcal{X}_C\}.$$

The RHS is the sheafification of the functor

$$S \mapsto \{S \to C, S^\#, \iota\}/\phi_C \times 1.$$

8. Relative Fargues-Fontaine curve

Let (R, R^+) be a perfectoid Huber pair in characteristic p. Define

$$\mathcal{Y}_{(0,\infty),(R,R^+)} := \mathrm{Spa}\, W(R^+) \setminus V(p[\varpi]),$$

where ϖ is a pseudo-uniformizer in R. The associated diamond satisfies

$$\mathcal{Y}^\diamond_{(0,\infty),(R,R^+)} \cong \mathrm{Spa}(R, R^+) \times (\mathrm{Spa}\,\mathbb{Q}_p)^\diamond.$$

Definition 8.1. The *relative Fargues-Fontaine curve* over $\mathrm{Spa}(R, R^+)$ is

$$\mathcal{X}_{(R,R^+)} := \mathcal{Y}_{(0,\infty),(R,R^+)}/\varphi^{\mathbb{Z}}.$$

The "diamond formula" is the same as before. In fact, if $S \in \mathrm{Perf}$ is arbitrary, you can define $\mathcal{X}_S$ by gluing $\mathcal{X}_{(R_i,R_i^+)}$ as $\mathrm{Spa}(R_i, R_i^+)$ runs over an affinoid perfectoid covering of S.

Warning 8.2. There is no map $X_S \to S$. (For instance S is of characteristic p while X_S is of characteristic 0.) However, there are shadows of such a map, e.g. there is a canonical map $|X_S| \to |S|$. (In fact, there is even a map $\mathcal{X}_{S,\text{ét}} \to S_{\text{ét}}$.)

CHAPTER 4

Fun with vector bundles on the Fargues-Fontaine curve Speaker: David Hansen

1. Vector bundles on the relative Fargues-Fontaine curve

Observe: the formulas that Jared wrote down to define the vector bundles $\mathcal{O}(d/n)$ make sense over any base, i.e. on any $\mathcal{X}_S$.

1.1. The scheme-theoretic curve.

DEFINITION 1.1. If (R, R^+) is some perfectoid Huber pair in characteristic p, define

$$X_R = \mathrm{Proj}\left(\bigoplus_{n\geq 0} H^0(\mathcal{X}_{\mathrm{Spa}(R,R^+)}, \mathcal{O}(n))\right)$$

WARNING 1.2. In general (if R is not a perfectoid field), this is not a noetherian scheme.

THEOREM 1.3 (GAGA). *There exists a functorial map of locally ringed spaces*

$$\mathcal{X}_{(R,R^+)} \to X_R$$

and pullback along this map induces an equivalence of categories of vector bundles.

REMARK 1.4. If R is not a perfectoid field, then it won't be Noetherian, and there isn't a good theory of coherent sheaves on $\mathcal{X}_{(R,R^+)}$.

1.2. Harder-Narasimhan filtration. If $(R, R^+) \leftrightarrow K$ is a perfectoid field, then X_K is Dedekind and complete, and every vector bundles has a Harder-Narasimhan filtration. In particular, after pulling back along $X_{\widehat{\overline{K}}} \to X_K$, any bundle $\mathcal{E}$ decomposes as a direct sum $\bigoplus \mathcal{O}(\frac{d_i}{n_i})$ for some uniquely determined $\frac{d_i}{n_i}$, counted with multiplicity.

REMARK 1.5. It is even true that the HN filtration is already split over K (although one doesn't have this classification result for bundles).

A book-keeping device: given $\mathcal{E}$ on $\mathcal{X}_K$, then after pulling back to $\mathcal{X}_{\widehat{\overline{K}}}$ we can write

$$\mathcal{E} \cong \bigoplus \mathcal{O}(\frac{d_i}{n_i}).$$

Define $\mathrm{HN}(\mathcal{E})$ to be the lower concave polygon joining $(0,0)$ and $(\mathrm{rank}\,\mathcal{E}, \deg \mathcal{E})$ having slopes $\frac{d_1}{n_1} \geq \frac{d_2}{n_2} \geq \ldots$ with horizontal lengths n_i.

Now suppose $\mathcal{E}$ is a vector bundle on some $\mathcal{X}_S$. For every point $s \in S$, the residue field $k(s)$ is perfectoid and so we get a map $\mathcal{X}_{(k(s),k(s)^+)} \to \mathcal{X}_S$. Using this, we can define a function

$$s \in S \mapsto \mathrm{HN}(\mathcal{E}_s).$$

THEOREM 1.6. *The function $s \mapsto \mathrm{HN}(\mathcal{E}_s)$ is upper semicontinuous (i.e. jumps up upon specializations).*

1.3. Moduli of vector bundles.

DEFINITION 1.7. Bun_n is the functor from Perf to groupoids sending any S to the groupoid of rank n vector bundles on $\mathcal{X}_S$.

The motivation for diamonds is to put enough geometric structure on Bun_n to make sense of ℓ-adic sheaves, etc.

2. Diamonds associated to cohomology of vector bundles on the curve

2.1. Motivating fact. Let X/k be a smooth projective connected curve, and let $\mathcal{E}$ be a vector bundle on X. Then the functor $\mathrm{Sch}/k \to \mathrm{Sets}$ sending $S \mapsto H^0(X \times_k S, \mathcal{E})$ is representable: it is just $\mathrm{Spec}\,\mathrm{Sym}_k\, H^0(X, \mathcal{E})^\vee$.

2.2. Is there an analogue of this in the Fargues-Fontaine setting?

DEFINITION 2.1. Fix $S \in \mathrm{Perf}$ and $\mathcal{E}$ a vector bundle on $\mathcal{X}_S$. Define $\mathcal{H}^i(\mathcal{E})\colon \mathrm{Perf}/S \to \mathrm{Sets}$ to be the sheafification of the presheaf sending $T \to S$ to $H^i(\mathcal{X}_T, \mathcal{E}_T)$.

REMARK 2.2. The presheaf $T \mapsto H^0(\mathcal{X}_T, \mathcal{E}_T)$ is already a sheaf.

2.2.1. *Basic facts:*

1. For all $i \geq 2$, $\mathcal{H}^i(\mathcal{E}) = 0$. (In the affinoid case, one can produce a cover of the relative FF curve by two affinoid subsets, so Cech cohomology vanishes.)
2. If $\mathrm{HN}(\mathcal{E}_s)$ has all slopes ≥ 0 for all $s \in S$, then $\mathcal{H}^1(\mathcal{E}) = 0$.
3. If $\mathrm{HN}(\mathcal{E}_s)$ has all slopes < 0, for all $s \in S$, then $\mathcal{H}^0(\mathcal{E}) = 0$.
4. If $\mathcal{E} = \mathcal{O}(n)$, then $\mathcal{H}^0(\mathcal{E})(R, R^+) = B(R, R^+)^{\varphi = p^n}$. When $n = 1$, this is the topological nilpotent elements, so it corresponds to an open ball.

THEOREM 2.3.

1. *For all $\mathcal{E}/\mathcal{X}_S$, $\mathcal{H}^0(\mathcal{E})$ is a locally spatial diamond, and $\mathcal{H}^0(\mathcal{E}) \to S$ is very well-behaved, e.g. it's smooth if all the slopes are > 0.*
2. *If $\mathrm{HN}(\mathcal{E}_s)$ has only negative slopes for all s, then $\mathcal{H}^1(\mathcal{E})$ is a locally spatial diamond.*

PROOF SKETCH. **Step 1:** For $\mathcal{E} = \mathcal{O}$, the diamond $\mathcal{H}^0(\mathcal{E})$ is simply $\underline{\mathbb{Q}_p}$. For $\mathcal{E} = \mathcal{O}(1)$, the diamond $\mathcal{H}^0(\mathcal{E})$ is representable: it is just an open ball in one variable.

Step 2: The case of $\mathcal{E} = \mathcal{O}(n)$ for $n \geq 1$. We need:

FACT 2.4. *$\mathcal{O}(n)$ can be realized as the cokernel of an injection $\mathcal{O}^{n-1} \hookrightarrow \mathcal{O}(1)^n$.*

(The idea is that $\mathcal{O}(1)$ has a lot of sections so the quotient is a vector bundle, and the classification theorem tells you which vector bundle.)

Now apply $\mathcal{H}^i$ to the SES

$$0 \to \mathcal{O}^{n-1} \to \mathcal{O}(1)^n \to \mathcal{O}(n) \to 0.$$

We get a SES in the category of abelian group sheaves on Perf $/S$

$$0 \to \mathcal{H}^0(\mathcal{O}^n) \to \mathcal{H}^0(\mathcal{O}(1)^n) \to \mathcal{H}^0(\mathcal{O}(n)) \to 0$$

because the slopes are such that $\mathcal{H}^1$ vanishes. The first map induces a free action (by translation) of $\mathcal{H}^0(\mathcal{O}^{n-1}) \simeq \underline{\mathbb{Q}_p}^{\oplus(n-1)}$ on $\mathcal{H}^0(\mathcal{O}(1)^n)$, a perfectoid ball in n variables. A free quotient of a locally spatial diamond by a locally profinite group is a locally spatial diamond.

Step 3 (Fargues): Write $\mathcal{E}$ as the kernel of some map $\mathcal{O}(n_1)^{d_1} \to \mathcal{O}(n_2)^{d_2}$. (To build this, find a quotient $\mathcal{O}(?)^? \twoheadrightarrow \mathcal{E}^\vee$ using that $\mathcal{O}(1)$ is "ample". Similarly find a surjection onto the kernel, and then dualize everything.) Then

$$\mathcal{H}^0(\mathcal{E}) \cong \mathcal{H}^0(\mathcal{O}(n_1)^{d_1}) \times_{\mathcal{H}^0(\mathcal{O}(n_2)^{d_2}),s} S$$

where $s\colon S \to \mathcal{H}^0(\mathcal{O}(n_2)^{d_2})$ is the zero-section.

Step 4: Next we consider the functor $\mathcal{H}^1$. If $\mathcal{E}/X_S$ is a vector bundle with all slopes ≥ 0 at all points, then (pro-étale locally on S) you can realize $\mathcal{E}$ as a quotient

$$0 \to \mathcal{O}(-1)^n \to \mathcal{O}^m \to \mathcal{E} \to 0.$$

Dualizing, one deduces that if $\mathcal{E}$ has all slopes < 0, then there exists a SES

$$0 \to \mathcal{E} \to \mathcal{O}^m \to \mathcal{O}(1)^n \to 0.$$

Passing to the LES, one finds

$$0 \to \underbrace{\mathcal{H}^0(\mathcal{E})}_{=0} \to \underline{\mathbb{Q}_p}^{\oplus m} \to \mathcal{H}^0(\mathcal{O}(1)^n) \to \mathcal{H}^1(\mathcal{E}) \to \underbrace{\mathcal{H}^1(\mathcal{O}^m)}_{=0}$$

Thus $\mathcal{H}^1(\mathcal{E}) \simeq \mathcal{H}^0(\mathcal{O}(1)^n)/\underline{\mathbb{Q}_p}^m$ is a (locally spatial) diamond. $\square$

2.3. More examples of diamonds.

DEFINITION 2.5. Fix two bundles $\mathcal{E}, \mathcal{F}$ on $\mathcal{X}_S$. Define

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F})\colon \mathrm{Perf} \to \mathrm{Sets}$$

to be the functor $T/S \mapsto \{\mathcal{O}_{\mathcal{X}_T} - \text{module maps } \mathcal{E}_T \to \mathcal{F}_T\}$. This is a locally spatial diamond over S, since

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \cong \mathcal{H}^0(\mathcal{E}^\vee \otimes_{\mathcal{O}_{\mathcal{X}_S}} \mathcal{F}).$$

(Furthermore, recall that $\mathcal{H}^0$ doesn't need to be sheafified.)

COROLLARY 2.6. *The diagonal map* $\Delta\colon \mathrm{Bun}_n \to \mathrm{Bun}_n \times \mathrm{Bun}_n$ *is "reasonable".*

Proof. Suppose given some $S \in \mathrm{Perf}$ and some map $S \to \mathrm{Bun}_n \times \mathrm{Bun}_n$, corresponding to two vector bundles $\mathcal{E}_1/\mathcal{X}_S$ and $\mathcal{E}_2/\mathcal{X}_S$. By defninition,

$$\mathrm{Bun}_n \times_{\Delta, \mathrm{Bun}_n \times \mathrm{Bun}_n} S$$

is the functor sending $T \to S$ to $\mathrm{Isom}(\mathcal{E}_1|_{\mathcal{X}_T}, \mathcal{E}_2|_{\mathcal{X}_T})$. This is a locally spatial diamond, e.g. because it's open inside $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$. $\square$

There are variants of this: $\mathrm{Surj}(\mathcal{E}, \mathcal{F})$ parametrizing surjective maps $\mathcal{E} \to \mathcal{F}$, and $\mathrm{Inj}(\mathcal{E}, \mathcal{F})$ parametrizing injective $\mathcal{E} \to \mathcal{F}$ which remain injective after any base change $\mathcal{X}_{S'} \to \mathcal{X}_S$.

These are both open inside $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$, hence they are locally spatial diamonds. The key point is that the map $|X_S| \to |S|$ is both open and closed.

3. Modification of vector bundles

Fact 3.1. *Given some perfectoid space S over $\mathbb{Q}_p$, consider $\mathcal{X}_{S^\flat}$. This has a canonical closed immersion $S \hookrightarrow \mathcal{X}_{S^\flat}$, which one can think of as a family of untilts of the residue fields of S.*

Definition 3.2. Given $S/\mathbb{Q}_p$, a *modification of vector bundles supported along* S is a pair of vector bundles $\mathcal{E}$ and $\mathcal{F}$ on $\mathcal{X}_{S^\flat}$ plus a "meromorphic" isomorphism

$$\mathcal{E}|_{\mathcal{X}_{S^\flat} - S} \xrightarrow{\sim} \mathcal{F}|_{\mathcal{X}_{S^\flat} - S}$$

Idea: moduli spaces of modifications of vector bundles on the Fargues-Fontaine curve are closely related to Rapoport-Zink spaces. On the other hand, this sort of structure appears in geometric Langlands.

3.1. The B_{dR}-affine Grassmannian.

Definition 3.3. Let $\mathrm{Gr}_{\mathrm{GL}_n}^{B_{\mathrm{dR}}}$ be the functor from $\mathrm{Perf}/(\mathrm{Spa}\,\mathbb{Q}_p)^\diamond$ [$=\mathrm{Perfd}/\mathbb{Q}_p$] to Sets which sends $S/\mathbb{Q}_p$ to the set of modifications

$$\mathcal{E}|_{\mathcal{X}_{S^\flat} - S} \xrightarrow{\sim} \mathcal{O}^n|_{\mathcal{X}_{S^\flat} - S}.$$

Theorem 3.4 (Scholze). *This functor is an inductive ind-(locally spatial) diamond.*

We will give some examples of modifications. Fix $S = \mathrm{Spa}\,\mathbb{C}_p$. Fix $\mathcal{E} = \mathcal{O}(2/5)$. Let's consider modifications $f\colon \mathcal{F} \hookrightarrow \mathcal{E}$ with the property that $\mathrm{coker}(f) \simeq i_{\infty *}\mathbb{C}_p^2$, where $i_\infty\colon \mathrm{Spa}\,\mathbb{C}_p \hookrightarrow \mathcal{X}_{\mathrm{Spa}\,\mathbb{C}_p^\flat}$. Which $\mathcal{F}$'s can occur?

It is immediate that $\deg \mathcal{F} = 0$ and $\mathrm{rank}\,\mathcal{F} = 5$. Also all slopes of $\mathcal{F}$ are $\leq 2/5$. A fun exercise shows that $\mathcal{F} \simeq \mathcal{O}^5$ or $\mathcal{F} \simeq \mathcal{O}(1/3) \oplus \mathcal{O}(-1/2)$.

By a form of Beauville-Laszlo, the space of all modifications of this type is $\mathrm{Gr}(2,5)$. Inside X there is some open subset $X^a \subset X$ parametrizing modifications with $\mathcal{F} \simeq \mathcal{O}^5$. (The a stands for "admissible".) In particular, X^a is an adic space over $\mathbb{Q}_p$.

What about the complement of X^a?

Fact 3.5. *The complement $X \setminus X^a$ is not an adic space. However $|X| - |X^a| \subset |X|$ is nice enough that it really does correspond to some subdiamond $X^{\diamond na} \subset X^\diamond$.*

REMARK 3.6. There is a notion of "canonical subdiamond" which is canonically associated to a nice enough subspace of the topological space of a diamond. The $X^{\diamond na}$ is even a "canonical subdiamond".

PROPOSITION 3.7. *$X^{\diamond na}$ is isomorphic to a punctured one-variable perfectoid ball over $\mathbb{C}_p$, modulo $D_{1/3}^{\times}$ where $D_{1/3}$ is the division algebra of invariant $1/3$ over $\mathbb{Q}_p$.*

Why is this true? The $X^{\diamond na}$ parametrizes modifications $\mathcal{F} \hookrightarrow \mathcal{O}(2/5)$ such that $\mathcal{F} \simeq \mathcal{O}(1/3) \oplus \mathcal{O}(-1/2)$ at all geometric points. By results of Kedlaya-Liu, there exists a canonical $\mathcal{F}^+$ such that $\mathcal{F}^+ \simeq \mathcal{O}(1/3)$ at all geometric points (the technical statement is that the Harder-Narasimhan filtration exists in families when it is constant pointwise). Then over $X^{\diamond na}$, one can define an $\mathrm{Aut}(\mathcal{O}(1/3))$-torsor by specifying a rigidification $r\colon \mathcal{O}(1/3) \simeq \mathcal{F}^+$. Composing f and r, we get a map $\widetilde{X^{\diamond na}} \to \mathrm{Inj}(\mathcal{O}(1/3), \mathcal{O}(2/5)) \subset \mathrm{Hom}(\mathcal{O}(1/3), \mathcal{O}(2/5)) \simeq \mathcal{H}^0(\mathcal{O}(1/15))$. Then use that $\mathrm{Aut}(\mathcal{O}(1/3))$ is $D_{1/3}^{\times}$.

Part 3

Day Three

CHAPTER 5

Bun_G for the Fargues-Fontaine curve
Speaker: Jared Weinstein

1. Some sheaves on Perf

Recall that Perf is the category of perfectoid spaces over $\mathbb{F}_p$, with the pro-étale topology.

We give some examples of sheaves of Perf that are of interest to us.

1.1. Perfectoid spaces. Let S be any perfectoid space. Then it represents a sheaf h_S.

1.2. The point. The final object $*$, sometimes written $(\operatorname{Spa}\mathbb{F}_p)^\diamond$, although this is not a diamond. It is an *absolute perfectoid space*, where we say X is an "absolute blah" if for all $S \in \mathrm{Perf}$, $X \times S$ is a "blah".

1.3. $(\operatorname{Spa}\mathbb{Q}_p)^\diamond$ is a diamond, with functor of points $S \mapsto \{S^\#/\mathbb{Q}_p\}$. To present it as a quotient of a perfectoid space by a pro-étale equivalence relation, we could write

$$(\operatorname{Spa}\mathbb{Q}_p)^\diamond = (\operatorname{Spa}\mathbb{Q}_p(\mu_{p^\infty})^{\wedge,\flat})/\mathbb{Z}_p^\times.$$

1.4. Analytic adic spaces. If $X/\mathbb{Z}_p$ is an analytic adic space, then $X^\diamond$ is an absolute diamond.

1.5. $(\operatorname{Spa}\mathbb{Z}_p)^\diamond$, with functor of points $S \mapsto \{S^\#\}$. This is an absolute diamond. For example, consider $(\operatorname{Spa}\mathbb{Z}_p)^\diamond \times \operatorname{Spa} C$ for an algebraically closed perfectoid field $C/\mathbb{F}_p$. This is $(\operatorname{Spa} W(\mathcal{O}_C) \setminus \{[\varpi] = 0\})^\diamond$, where ϖ is a pseudo-uniformizer of C, and $\operatorname{Spa} W(\mathcal{O}_C) \setminus \{[\varpi] = 0\}$ is an analytic adic space.

1.6. Constant spaces.

1.6.1. *Finite sets.* Let T be a finite set. Then $\underline{T}(S) = \mathrm{Hom}_{\mathrm{top}}(|S|, T)$, so $\underline{T} = T \times *$. This is an absolute perfectoid space.

1.6.2. *Profinite sets.* More generally, let $T = \varprojlim T_i$ be a profinite set, $\underline{T}(S) = \mathrm{Hom}_{\mathrm{top}}(|S|, T)$ is an absolute perfectoid space, e.g. $\operatorname{Spa} C \times \underline{T} = \operatorname{Spa} R$, where $R = \mathrm{Hom}_{\mathrm{top}}(T, C)$. We claim that this is a perfectoid algebra, with $R^\circ = \mathrm{Hom}_{\mathrm{top}}(T, \mathcal{O}_C)$. Note that $|\operatorname{Spa} R| = T$. (The topology on R is the topology of uniform convergence; we can even make R a Banach algebra by giving it the sup norm.)

1.6.3. *Locally profinite spaces.* More generally, one could even allow T to be locally profinite.

1.7. Cohomology of bundles over the Fargues-Fontaine curve. Let M be an isocrystal over $\overline{\mathbb{F}}_p$. We described how to construct from this a vector bundle $\mathcal{E}_M$ on "the" Fargues-Fontaine curve over any S for $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$. There is a diamond

$$\mathcal{H}^i(\mathcal{E}_M)\colon S \mapsto H^i(\mathcal{X}_S, \mathcal{E}_M).$$

This is an absolute diamond, and even an absolute perfectoid space if the slopes of $\mathcal{E}_M$ lie in $[0,1]$.

Example 1.1. $H^0(\mathcal{O}_{\mathcal{X}}) = \underline{\mathbb{Q}_p}$.

Example 1.2. $H^0(\mathcal{O}(1))(\mathrm{Spa}\, R) = B_R^{\phi=p} = 1 + R^{00}$. This is a $\mathbb{Q}_p$-vector space. So we see that $H^0(\mathcal{O}(1))$ is represented by $\mathrm{Spa}\, \overline{\mathbb{F}}_p[[t^{1/p^\infty}]]^\diamond$, an absolute perfectoid space. For example,

$$\mathrm{Spa}\, C \times_{\overline{\mathbb{F}}_p} \mathrm{Spa}\, \overline{\mathbb{F}}_p[[t^{1/p^\infty}]] = \mathrm{Spa}\, \mathcal{O}_C[[t^{1/p^\infty}]] \setminus \{\varpi = 0\}.$$

This is a perfectoid space; informally it is the "perfectoid open ball over C".

Example 1.3. $\mathcal{H}^1(\mathcal{O}(-1))(S)$ is the "sheafification of $S \mapsto R^\#/\underline{\mathbb{Q}_p}$ if $S = \mathrm{Spa}\, R$ and $R^\#$ is some untilt," or more precisely,

$$\mathcal{H}^1(\mathcal{O}(-1)) \times (\mathrm{Spa}\, \mathbb{Q}_p)^\diamond \simeq (\mathbb{A}^1_{\mathbb{Q}_p})^\diamond / \underline{\mathbb{Q}_p}.$$

(The diamond $\mathcal{H}^1(\mathcal{O}(-1))$ is sort of similar to the algebraic space $\mathbb{A}^1/\underline{\mathbb{Z}}$.)

2. Some stacks on Perf

2.1. $B\underline{G}$**.** Let G be a locally profinite topological group. Then there is a stack $B\underline{G} = [*/\underline{G}]$, with morphisms $S \to [*/\underline{G}]$ classifying pro-étale $\underline{G}$-torsors on S.

For example, $[*/\underline{\mathrm{GL}_n(\mathbb{Q}_p)}]$ classifies pro-étale $\underline{\mathbb{Q}_p}$-local systems of rank n.

2.2. Where are we? In classical geometric Langlands, one considers for $\mathbb{F}_q$-schemes S the product $X \times_{\mathbb{F}_q} S$ where $X/\mathbb{F}_q$ is a curve.

Now, we're considering for $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ the relative curve $\mathcal{X}_S$, which *doesn't* have a map to S. Note however that $\mathcal{X}_S^\diamond \simeq \mathrm{Spa}\, \mathbb{Q}_p^\diamond \times S/(1 \times \phi_S^{\mathbb{Z}})$.

2.3. Bun_n**.** $\mathrm{Bun}_n(S)$ is the groupoid of rank n vector bundles on $X \times_{\mathbb{F}_q} S$ (in the classical case) or $\mathcal{X}_S$ (in the Fargues-Fontaine case).

The points of Bun_n are

$$|\mathrm{Bun}_n| \simeq \begin{cases} \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/\prod_v \mathrm{GL}_n(\mathcal{O}_v) & \text{classical} \\ \{\text{isocrystals } /\overline{\mathbb{F}}_p\} & \text{Fargues-Fontaine} \end{cases}$$

Recall that all isocrystals over $\overline{\mathbb{F}}_p$ are of the form $\bigoplus \mathcal{O}(d_i/n_i)$ with $\sum n_i = n$. The topology can be described by HN polygons. In specializations, the HN polygons go up.

EXAMPLE 2.1. Over $\mathbb{P}^1$, rank 2 bundles specialize as $\mathcal{O} \oplus \mathcal{O} \rightsquigarrow \mathcal{O}(1) \oplus \mathcal{O}(-1) \rightsquigarrow \mathcal{O}(2) \oplus \mathcal{O}(-2) \rightsquigarrow \dots$.

Over $\mathcal{X}_C$, the exact same holds. For something more interesting, let's look at $\mathcal{O}(1/2)$. Then we mentioned that $\mathcal{O}(1/2)$ specializes to $\mathcal{O} \oplus \mathcal{O}(1)$, for example. What this means more precisely is that for the relative curve $\mathcal{X}_S$, one can find families of vector bundles in which the isomorphism types along fibers specialize in this way.

In particular, $\mathrm{Bun}_n^{\mathrm{ss}}$ is open (and dense) in Bun_n. The words semistable, isoclinic, and basic are synonymous: they all mean that there is only one slope.

Then $\mathrm{Bun}_n^{\mathrm{ss}} = \coprod_{d \in \mathbb{Z}} \mathrm{Bun}_n^{\mathrm{ss}, \deg d}$. What is $\mathrm{Bun}_n^{\mathrm{ss}, \deg 0}$?

THEOREM 2.2 (Kedlaya-Liu). *The following categories are equivalent:*

(1) $\underline{\mathbb{Q}_p}$*-local systems of rank* n *on* S.

(2) $\mathrm{Bun}_n^{\mathrm{ss}, \deg 0}(S)$. *In other words,* $\mathrm{Bun}_n^{\mathrm{ss}, \deg 0} = [*/\underline{\mathrm{GL}_n(\mathbb{Q}_p)}]$.

The map from (2) to (1) takes $\mathcal{E}$ to $\mathbb{L}$, where $\mathbb{L}(T \to S) = H^0(\mathcal{X}_T, \mathcal{E})$.

EXAMPLE 2.3. Consider $\mathrm{Bun}_2^{\mathrm{ss}, \deg 1}$. The only point is $\mathcal{O}(1/2)$, which corresponds to the matrix

$$\phi_M = \begin{pmatrix} 0 & 1 \\ p^{-1} & 0 \end{pmatrix}.$$

Then $\mathrm{End}_{\phi_M}(M) \subset M_2(W(\overline{\mathbb{F}}_p)[1/p])$ can be identified as the non-split quaternion algebra $D/\mathbb{Q}_p$. So the conclusion is that $\mathrm{Bun}_2^{\mathrm{ss}, \deg 1} \simeq [*/D^\times]$. There is an open subset $\mathrm{Bun}_2^{\deg 1, \leq 1}$ in which only these bundles appear. There is a specialization $\mathcal{O}(1/2) \rightsquigarrow \mathcal{O}(1) \oplus \mathcal{O}$ in $\mathrm{Bun}_2^{\deg 1, \leq 1}$, and the latter has the stratification

$$[*/\underline{D}^\times] \subset \mathrm{Bun}_2^{\deg 1, \leq 1} \supset [*/\underline{\mathrm{Aut}}(\mathcal{O} \oplus \mathcal{O}(1))]$$

where $\underline{\mathrm{Aut}}(\mathcal{E})$ is the sheaf $S \mapsto \mathrm{Aut}_{\mathcal{X}_S}(\mathcal{O} \oplus \mathcal{O}(1))$.

Note that $\mathcal{H}^1(\mathcal{O}(-1)) \simeq \underline{\mathrm{Ext}}(\mathcal{O}, \mathcal{O}(-1)) \simeq \underline{\mathrm{Ext}}(\mathcal{O}(1), \mathcal{O})$ classifies exact sequence

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0.$$

Note that $\mathcal{E} \in \mathrm{Bun}_2^{\deg \leq 1}$. This can be used to produce a uniformization of $\mathrm{Bun}_2^{\deg \leq 1}$.

2.4. Hecke stacks. We want to define a Hecke correspondence

$$\begin{tikzcd} & \mathrm{Hecke} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \times \mathrm{Spa}\, \mathbb{Q}_p^\diamond \end{tikzcd}$$

We define

$$\mathrm{Hecke}(S) = \left\{ (\mathcal{E}_1, \mathcal{E}_2, S^\#, \text{ meromorphic } \mathcal{E}_1|_{\mathcal{X}_S - S^\#} \xrightarrow{\sim} \mathcal{E}_2|_{\mathcal{X}_S - S^\#} \right\}.$$

(Note that $S^\#$ is an effective Cartier divisor of degree 1 in $\mathcal{X}_S$.)

There is a substack $\mathrm{Hecke}^{\mathrm{elem}, d}$ classifying

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to i_{S^\# *} W \to 0$$

where $i_{S^\#} : S^\# \hookrightarrow \mathcal{X}_S$ where W is an $\mathcal{O}_{S^\#}$-module locally free of rank d.

THEOREM 2.4 (Scholze-W.). *Let $S = C$, and choose $C^\#/\mathbb{Q}_p$. The following are in bijection:*

(1) The isogeny classes of p-divisible groups over $\mathcal{O}_{C^\#}$.
(2) Isomorphism classes of elementary modifications $0 \to \mathcal{F} \to \mathcal{E} \to i_{C^\#*}W \to 0$ where $\mathcal{F}$ is semi-simple of degree 0.

PROOF. (1) $\implies$ (2). Start with $G/\mathcal{O}_{C^\#}$ (which you may assume is formal). Then G is associated to a power series in $2d$ variables $X + Y = \dots$. So $G(\mathcal{O}_{C^\#}) = \mathfrak{m}^d_{\mathcal{O}_{C^\#}}$. We get an exact sequence of $\mathbb{Z}_p$-modules

$$0 \longrightarrow G[p^\infty](\mathcal{O}_{C^\#}) \longrightarrow G(\mathcal{O}_{C^\#}) \xrightarrow{\log_G} \operatorname{Lie} G[1/p] \longrightarrow 0.$$

We're going to convert this to an exact sequence of $\mathbb{Q}_p$-vector spaces by applying $\varprojlim_p$. We get

$$0 \longrightarrow VG \longrightarrow \widetilde{G}(\mathcal{O}_{C^\#}) \xrightarrow{\log_G} \operatorname{Lie} G[1/p] \longrightarrow 0 \tag{2.1}$$

This is an extension of a $\mathbb{C}_p$-vector space by a $\mathbb{Q}_p$-vector space, i.e. a "Banach-Colmez space".

The map $\widetilde{G}(\mathcal{O}_{C^\#}) \to \widetilde{G}(\mathcal{O}_{C^\#}/p)$ is an isomorphism (perhaps surprisingly?). To get the inverse, you send a sequence $(\overline{x}_0, \overline{x}_1, \dots) \in \widetilde{G}(\mathcal{O}_{C^\#}(p))$ to $(y_0, y_1, \dots)$ where $y_i = \lim_{j\to\infty}[p^j](x_{i+j})$.

We also need to use a theorem that G is isotrivial, i.e. there exists a quasi-isogeny $\rho\colon G \otimes_{\mathcal{O}_C^\#} \mathcal{O}_{C^\#/p} \to G_0 \otimes_{\overline{\mathbb{F}}_p} \mathcal{O}_{C^\#}/p$ where $G_0/\overline{\mathbb{F}}_p$ is a p-divisible group.

Let $\mathcal{E}_{G_0}/\mathcal{X}_C$ be the corresponding vector bundle. Then it's a fact that $H^0(\mathcal{X}_C, \mathcal{E}_{G_0}) = \widetilde{G}_0(\mathcal{O}_C)$. We have the chain of identifications

$$\widetilde{G}(\mathcal{O}_{C^\#}) \xrightarrow{\sim} \widetilde{G}(\mathcal{O}_{C^\#}/p) \xrightarrow{\sim} \widetilde{G}_0(\mathcal{O}_{C^\#}/p) \simeq \widetilde{G}_0(\mathcal{O}_C/\varpi) \simeq \widetilde{G}_0(\mathcal{O}_C) \simeq H^0(\mathcal{X}_C, \mathcal{E}_{G_0}).$$

Putting this into (2.1), we get a SES of vector bundles

$$0 \to VG \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{X}_C} \to \mathcal{E}_{G_0} \to i_{C^\#*} \operatorname{Lie} G[1/p] \to 0.$$

(One can see that this is a modification outside $C^\#$.)

$\square$

3. Connection to Rapoport-Zink spaces

Let $G_0/\overline{\mathbb{F}}_p$ be a fixed p-divisible group of dimension d. We define a functor M_{G_0} from complete $W(\overline{\mathbb{F}}_p)$-algebras to sets, sending

$$R \mapsto \left\{ (G, \rho) \mid \rho\colon G \otimes_R R/p \to G_0 \otimes_{\overline{\mathbb{F}}_p} R/p \text{ a quasi-isogeny} \right\}.$$

THEOREM 3.1 (Rapoport-Zink). *M_{G_0} is a formal scheme (locally formally of finite type over $W(\overline{\mathbb{F}}_p)$), which locally admits a finitely generated ideal of definition.*

Hence we can take the "adic generic fiber" $\mathcal{M}_{G_0} := (M^{\mathrm{ad}}_{G_0})_\eta$. Here, we applied the functor from formal schemes over $\operatorname{Spf} W(\overline{\mathbb{F}}_p)$ to adic spaces over $\operatorname{Spa} W(\overline{\mathbb{F}}_p)$.

Then we take the diamond to get $\mathcal{M}^{\diamond}_{G_0} \to \mathrm{Spa}(W(\overline{\mathbb{F}}_p)[1/p])^{\diamond}$. Here $\mathcal{M}^{\diamond}_{G_0}$ sends $S/W(\overline{\mathbb{F}}_p)[1/p]$ to the set of elementary modifications of degree d:

$$\mathcal{F} \to \mathcal{E}_{G_0}$$

over $\mathcal{X}_{S^\flat}$ occurring over S, such that $\mathcal{F}$ is semi-stable of degree 0, together with a $\mathbb{Z}_p$-lattice $\mathbb{L}_0 \subset \mathbb{L}$ where $\mathbb{L}$ is the $\mathbb{Q}_p$-local system corresponding to $\mathcal{F}$ (by Kedlaya-Liu).

We can pass to infinite level:

$$\begin{tikzcd} \mathcal{M}^{\diamond}_{G_0,\infty} \arrow[r, equal] \arrow[d] & \{ \text{ elementary modifications } \mathcal{O}^n \to \mathcal{E}_{G_0,L} \ldots \} \\ \mathcal{M}^{\diamond}_{G_0} & \end{tikzcd}$$

The object $\mathcal{M}^{\diamond}_{G_0,\infty}$ comes from a perfectoid space.

$$\begin{tikzcd} & \mathcal{M}^{\diamond}_{G_0,\infty} \arrow[dl] \arrow[d] \arrow[dr] & \\ * \arrow[dd, "\mathcal{O}^n"] & \arrow[d] & * \times \mathrm{Spa}(W(\overline{\mathbb{F}}_p)[1/p]) \arrow[dd, "\mathcal{E}_{G_0}"] \\ & \mathrm{Hecke}_e^{\mathrm{elem},d} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \times \mathrm{Spa}(W(\overline{\mathbb{F}}_p)[1/p]) \end{tikzcd}$$

CHAPTER 6

Local class field theory and the Fargues-Fontaine curve Speaker: Arthur-César le Bras

1. Geometric class field theory (unramified)

Let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$, and let $K = \mathbb{F}_q(X)$.

Let $\mathbb{A} = \prod'_{x\in|X|} \widehat{K}_x$, $\mathcal{O} = \prod_{x\in|X|} \widehat{\mathcal{O}}_x$. Unramified class field theory says that

$$\left(K^\times\backslash\mathbb{A}^\times/\mathcal{O}^\times\right)^\wedge \cong (\mathrm{Gal}_K^{\mathrm{unr}})^{\mathrm{ab}}$$

via the map

$$(a_x)_{x\in|X|} \mapsto \prod_{x\in|X|} \mathrm{Frob}_x^{\mathrm{ord}_x(a_x)}.$$

We want to reformulate this statement more geometrically, in terms of X. Observe first that

- $\mathrm{Gal}_K^{\mathrm{unr}} \cong \pi_1(X)$.
- $K^\times\backslash\mathbb{A}^\times/\mathcal{O}^\times \simeq \mathrm{Pic}_X(\mathbb{F}_q)$.

The geometric reformulation is: there is a natural bijection

$$\left\{\begin{matrix}\text{continuous characters}\\ \pi_1(X)\to\overline{\mathbb{Z}}_\ell^\times\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{continuous characters}\\ \mathrm{Pic}_X(\mathbb{F}_q)\to\overline{\mathbb{Z}}_\ell^\times\end{matrix}\right\}$$

sending $\rho\mapsto\chi_\rho$ for all $x\in|X|$, such that $\rho(\mathrm{Frob}_x) = \chi_\rho(\mathcal{O}(x))$.

One can go further, thanks to the following:

- a continuous character $\pi_1(X) \to \overline{\mathbb{Z}_\ell}^\times$ is equivalent to a "rank 1 $\overline{\mathbb{Z}}_\ell$-local system on X".
- a continuous character $\mathrm{Pic}_X(\mathbb{F}_q) \to \overline{\mathbb{Z}_\ell}^\times$ is equivalent to a *character sheaf*, that is to a rank 1 $\overline{\mathbb{Z}}_\ell$-local system $\mathcal{F}$ on Pic_X such that $m^*\mathcal{F} \simeq p_1^*\mathcal{F}\otimes p_2^*\mathcal{F}$, where $m\colon \mathrm{Pic}_X\times\mathrm{Pic}_X\to\mathrm{Pic}_X$ is the multiplication and p_1, p_2 are the two projections.

  Indeed, given a character sheaf one takes the trace function. Conversely, use the Lang isogeny to go the other way.

There is an Abel-Jacobi morphism $AJ\colon X\to\mathrm{Pic}(X)$ sending $x\in X$ to $\mathcal{O}(x)$. The geometric reformulation says: the pullback functor AJ^* induces an equivalence of categories

$$\{\text{character sheaves on } \mathrm{Pic}_X\} \xrightarrow{\sim} \{\text{rank 1 } \overline{\mathbb{Z}}_\ell\text{-local systems on } X\}.$$

Note in passing that, contrary to the original statement, the statement now makes sense over any base field.

Deligne. let $\mathcal{F}$ be a rank 1 $\overline{\mathbb{Z}}_\ell$-local system on X. Let $d \geq$, and $X^{(d)} = X^d/S_d$, the "moduli space of degree d effective Cartier divisors on X".

Set $\mathcal{F}^{(d)} = (\pi_* \mathcal{F}^{\boxtimes d})^{S_d}$ where $\pi \colon X^d \to X^{(d)}$.

Fact 1.1. *The sheaf $\mathcal{F}^{(d)}$ is again a local system on $X^{(d)}$.*

Indeed, the sheaf $\mathcal{F}^{\boxtimes d}$ tautologically descends to the stacky quotient $[X^d/S_d]$, and one sees using that $\mathcal{F}$ has rank 1 that the action of the stabilizers act trivially on the stalks of the sheaf obtained on $[X^d/S_d]$ (in general, $\mathcal{F}^{(d)}$ would only be a perverse sheaf).

We have $AJ^d \colon X^{(d)} \to \mathrm{Pic}^d_X$ sending $D \mapsto \mathcal{O}(D)$. The old AJ coincides with AJ^1.

The fiber of AJ^d above $\mathcal{O}(D)$ is the linear system $|D| = \mathbb{P}(H^0(X, \mathcal{O}(D)))$. By Riemann-Roch, if $d > 2g-2$ then AJ^d is a fibration in projective spaces.

Fact 1.2. *For all $n \geq 1$, $\pi_1(\mathbb{P}^n_{\overline{\mathbb{F}}_q}) = 0$.*

For all $d > 2g-2$, $\pi_1(\mathrm{Pic}^d_X) = \pi_1(X^{(d)})$. So $\mathcal{F}^{(d)}$ descends to a local system on Pic^d_X, called $\mathcal{A}^d_{\mathcal{F}}$. Let $\mu \colon X \times \mathrm{Pic}^d_X \to \mathrm{Pic}^{d+1}_X$ by $(x, \mathscr{L}) \mapsto \mathscr{L}(x)$. Then $\mu^* \mathcal{A}^{d+1}_{\mathcal{F}} \simeq \mathcal{F} \boxtimes \mathcal{A}^d_{\mathcal{F}}$ for all $d > 2g-2$. This allows to extend $\mathcal{A}^d_{\mathcal{F}}$ to all d (if one fixes a rational point of X), and to check the character sheaf property. □

Remark 1.3. We could have worked with the Picard stack instead of the Picard scheme. As the former is a $\mathbb{G}_m$-gerb over the latter and as $\mathbb{G}_m$ is connected, this makes no difference when working with ℓ-adic local systems. We will see below that in the context of local class field theory, the difference becomes important.

Remark 1.4. One can also recover ramified class field theory in this way. Serre's book explains an approach of Lang-Rosenlicht via generalized Jacobians. Recent work of Guignard gives a different argument, more along the lines of Deligne's for the unramified case.

2. The Fargues-Fontaine curve

In 2016, Fargues gave a geometric reformulation/proof of local (ramified !) class field theory.

Let E be a local field, π a uniformizer of E, and $\mathbb{F}_q$ the residue field.

How do you attach an interesting geometric object to E, whose fundamental group is $\mathrm{Gal}(\overline{E}/E)$?

We will sketch Fargues' proof for the case $E = \mathbb{F}_q((\pi))$, although the positive characteristic assumption doesn't get used until the end. Recall that if T is a scheme of finite type over $\mathbb{F}_q$, and $F \supset \mathbb{F}_q$ is an algebraically closed field, then there is an equivalence

$$\{\text{finite étale covers of } T\} \leftrightarrow \left\{ \begin{matrix} \text{finite étale covers } T' \to T \times_{\mathbb{F}_q} F \\ \varphi_F^* T' \xrightarrow{\sim} T', \varphi_F = \mathrm{Id} \times \mathrm{Frob}_F \end{matrix} \right\}$$

Fargues-Fontaine realized that, when trying to do geometry over "$T = \mathrm{Spec}(E)$", it is better to replace F by C, an algebraically closed perfectoid field containing $\mathbb{F}_q$. Then instead of forming the fibered product $T \times_{\mathbb{F}_q} F$ in schemes, consider $\mathcal{Y}_{C,E} = \mathrm{Spa}\, E \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}\, C$. Then define $\mathcal{X}_{C,E} = \mathcal{Y}_{C,E}/\varphi_C^{\mathbb{Z}}$. One has

$$\mathcal{O}(\mathcal{Y}_{C,E}) = \left\{ \sum_{n\in\mathbb{Z}} x_n \pi^n \colon x_n \in C, \lim_{|n|\to\infty} |x_n|\rho^n = 0 \text{ for all } \rho \in (0,1) \right\}.$$

In general, for any E (not necessarily of positive characteristic) and C as before, define

$$\mathcal{Y}_{C,E} = \mathrm{Spa}(W_{\mathcal{O}_E}(\mathcal{O}_C)) \setminus V(\pi[\varpi]),$$

where $W_{\mathcal{O}_E}(-) = W(-) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$. Then define

$$\mathcal{X}_{C,E} = \mathcal{Y}_{C,E}/\varphi_C^{\mathbb{Z}}.$$

A clue that this is going in the right direction is given by the following result.

Fact 2.1. *We have $\pi_1(\mathcal{X}_{C,E}) = \mathrm{Gal}(\overline{E}/E)$.*

Proof. We want to check that pullback induces an equivalence between finite étale E-algebras, and finite étale $\mathcal{O}_{\mathcal{X}_{C,E}}$-algebras. Let $\mathcal{E}$ be a finite $\mathcal{O}_X$-algebra. The classification theorem says that $\mathcal{E} = \bigoplus \mathcal{O}_{\mathcal{X}_{C,E}}(\lambda_i)$ for $\lambda_i \in \mathbb{Q}$ as a vector bundle. We have a non-degenerate trace pairing $\mathcal{E} \otimes \mathcal{E} \to \mathcal{O}_{\mathcal{X}_{C,E}}$ which implies $\sum \lambda_i = 0$. Let $\lambda = \max(\lambda_i)$.

The non-degenerate map $\mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ restricts to a non-zero map $\mathcal{O}(\lambda) \otimes \mathcal{O}(\lambda) \to \mathcal{E}$ which forces $\lambda \leq 0$ (since vector bundles with negative slopes have no global sections). The symmetry of the trace pairing then shows that $\lambda_i = 0$ for all i. Then $\mathcal{E} = A \otimes_E \mathcal{O}_{\mathcal{X}_{C,E}}$, for some étale E-algebra A. $\square$

Nevertheless, as we will see below, the Fargues-Fontaine curve $\mathcal{X}_{C,E}$ is not exactly the object which will play the role of the curve X of geometric class field theory. This has to do with the fact that the correct way of putting the Fargues-Fontaine curve in families is not the obvious one (by taking fiber products over $\mathrm{Spa}(E)$, with test objects being adic spaces over E). Indeed, we have seen that if $S \in \mathrm{Perf}_{\mathbb{F}_q}$, one can define $\mathcal{Y}_{S,E}$ and $\mathcal{X}_{S,E}$ (the definition of the previous lectures given for $E = \mathbb{Q}_p$ immediately adapts to any E).

Warning 2.2. $\mathcal{Y}_{S,E}$ and $\mathcal{X}_{S,E}$ do not live over S, even if $E = \mathbb{F}_q((\pi))$!

3. Diamonds

To get a geometric reformulation of local CFT, we need to work in in an even bigger category than the category of adic spaces, namely the category of sheaves on $\mathrm{Perf}_{\mathbb{F}_q,\text{pro-étale}}$.

Example 3.1. If S is an adic space over $\mathbb{F}_q$, then h_S is a sheaf on $\mathrm{Perf}_{\mathbb{F}_q,\text{pro-étale}}$.

Any analytic adic space Z over $\mathbb{Z}_p$ induces a sheaf $Z^\diamond$ on $\mathrm{Perf}_{\mathbb{F}_q,\text{pro-étale}}$, whose points are

$$S \mapsto \{\text{untilts } (S^\#, i) \text{ of } S/Z\}/\sim .$$

If S is perfectoid, $Z^\diamond = (Z^\flat)^\diamond$. This sheaf is even a diamond, i.e. the analogue of an algebraic space for the pro-étale topology.

EXAMPLE 3.2. Let T be a topological space. The functor

$$S \mapsto C(|S|, T)$$

is a pro-étale sheaf, since a surjective pro-étale morphism is a quotient map on the underlying topological spaces. If T is locally profinite, then this is even an absolute diamond.

REMARK 3.3. As sheaves on $\mathrm{Perf}_{\mathbb{F}_q,\text{pro-étale}}$,

$$Y_{S,E}^\diamond = \mathrm{Spa}(E)^\diamond \times_{\mathrm{Spa}\,\mathbb{F}_q} S$$

and

$$X_{S,E}^\diamond = \mathrm{Spa}(E)^\diamond \times_{\mathrm{Spa}\,\mathbb{F}_q} S/(1 \times \varphi_S).$$

This gives in the world of diamonds a precise meaning to the heuristic used above to motivate the introduction of the curve.

4. The Abel-Jacobi morphism for local fields

We can now define all the geometric objects involved in the geometric reformulation of local CFT.

4.1. Picard stacks. Let Pic be the functor on $\mathrm{Perf}_{\mathbb{F}_q}$ sending S to the groupoid of line bundles on $\mathcal{X}_{S,E}$. Here is a special case of a more general result of Kedlaya-Liu.

PROPOSITION 4.1. $\mathrm{Pic} = \coprod_{d\in\mathbb{Z}} \mathrm{Pic}^d$ *and for all* $d \in \mathbb{Z}$,

$$\mathrm{Pic}^d \simeq [\mathrm{Spa}\,\mathbb{F}_q/\underline{E}^\times],$$

the classifying stack of $\underline{E}^\times$*-torsors. The identification is* $\mathscr{L} \mapsto \mathrm{Isom}(\mathcal{O}(d), \mathscr{L})$. *Hence* Pic *is a stack on* Perf.

This is saying that if $S \in \mathrm{Perf}_{\mathbb{F}_q}$, and $\mathscr{L}$ is the line bundle of degree d on $\mathcal{X}_{S,E}$ then there exists a pro-étale cover $S' \to S$ such that $\mathscr{L}|_{\mathcal{X}_{S',E}} \simeq \mathcal{O}(d)$, and $\mathrm{Aut}(\mathcal{O}(d)) = \underline{E}^\times$ (here one really sees that the relative Fargues-Fontaine curve is far from being a product !).

COROLLARY 4.2. *There is an equivalence between*

$$\{\overline{\mathbb{Q}}_\ell - \textit{local systems on } \mathrm{Pic}^d_{\mathbb{F}_q}\} \leftrightarrow \{\textit{continuous } \overline{\mathbb{Q}}_\ell\textit{-representations of } E^\times\}$$

4.2. Divisors. Let $d \geq 1$, and Div^d be the moduli of effective Cartier divisors[1] of degree d on the curve. This has functor of points

$$S \mapsto \left\{ (\mathscr{L}, u) \colon \begin{array}{c} \mathscr{L} = \text{ line bundle of degree } d \text{ on } \mathcal{X}_{S,E} \\ u \in H^0(\mathcal{X}_{S,E}, \mathscr{L}) \text{ such that for all } s \in S,\ u|_{X_{k(s)}} \neq 0 \end{array} \right\}$$

Then Div^d has 2 different descriptions:

[1] A notion which we won't define precisely in this talk.

(1) $\mathrm{Div}^d \simeq [(B_E^{\varphi=\pi^d} \setminus \{0\})/\underline{E}^\times]$ where $B_E^{\varphi=\pi^d}$ is the sheaf $S \mapsto H^0(\mathcal{X}_{S,E}, \mathcal{O}(d))$. This is easily deduced from the definition of Div^d and the description of Pic^d given before.

Via these identifications, the Abel-Jacobi morphism $(\mathscr{L}, u) \mapsto \mathscr{L}$ over $\overline{\mathbb{F}}_p$ is the natural morphism $[(B_{E,\overline{\mathbb{F}}_p}^{\varphi=\pi^d} \setminus \{0\})/\underline{E}^\times] \to [\mathrm{Spa}\,\overline{\mathbb{F}}_p/\underline{E}^\times]$. This is a fibration with fiber $B_{E,\overline{\mathbb{F}}_p}^{\varphi=\pi^d} \setminus \{0\}$.

(2) $\mathrm{Div}^1 \simeq \mathrm{Spa}(E)^\diamond/\varphi_{E^\diamond}$. Here $\varphi_{E^\diamond}$ is the Frobenius on any pro-étale sheaf on Perf.

Let us explain how this identification works. Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_p}$. An S-point of $\mathrm{Spa}(E)^\diamond$ is equivalent to an untilt $(S^\#, i)$ of S over E. We have a bijection

$$\{\text{untilts of } R \text{ over } E\} \leftrightarrow \left\{ \begin{matrix} \deg 1 \text{ primitive elements} \\ \sum_{n\geq 0}[x_n]\pi^n \in W_{\mathcal{O}_E}(R^\circ) \\ x_0 \in R^\times \cap R^{00}, x_1 \in (R^\circ)^\times \end{matrix} \right\} / W_{\mathcal{O}_E}(R^\circ)^\times.$$

The map $\leftarrow$ is: given ξ on the RHS, form $R^\# = W_{\mathcal{O}_E}(R^\circ)[1/\pi]/(\xi)$. The map $\to$ is: $R^\# \mapsto \ker(\theta_{R^\#} : W_{\mathcal{O}_E}(R^\circ) \to R^{\#\circ})$.

We have $V(\xi) = \mathrm{Spa}(R^\#, R^{+\#}) \hookrightarrow \mathcal{Y}_{S,E}$. This we have map from $\mathrm{Spa}(E)^\diamond$ to Cartier divisors D on $\mathcal{Y}$ which takes $\xi \mapsto V(\xi)$. This then maps to Div^1 via $D \mapsto \sum_{n\in\mathbb{Z}} \varphi^n(D)$, and the composite factors through $\mathrm{Spa}(E)^\diamond/\varphi_{E^\diamond}$, and the claim is that the induced map is an isomorphism.

Using this second identification, we get

- Div^1 is a diamond, with an étale presentation given by $\mathrm{Spa}(E)^\diamond \to \mathrm{Spa}(E)^\diamond/\varphi_{E^\diamond}$ (note that the first identification only shows a priori that Div^1 is an absolute diamond). In particular, one deduces that $B_E^{\varphi=\pi^d} \setminus \{0\}$ is a diamond, since it is pro-étale over it (again, this is not obvious a priori, since $B_E^{\varphi=\pi^d}$ is only an absolute diamond). The diamond Div^1 is a quite exotic one : it is not quasi-separated, although the map $\mathrm{Div}^1 \to \mathrm{Spa}(\mathbb{F}_q)$ is separated (even, in a precise sense, proper and smooth).
- A $\overline{\mathbb{Q}}_\ell$-local system on $\mathrm{Div}^1_{\overline{\mathbb{F}}_q}$ is equivalent to a $\overline{\mathbb{Q}}_\ell$-local system on $(\mathrm{Spa}(E)^\diamond \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\,\overline{\mathbb{F}}_q)/\varphi_{\overline{\mathbb{F}}_q}$ (we can switch the Frobenius because $\varphi_{E^\diamond} \times \varphi_{\overline{\mathbb{F}}_q}$ is the absolute Frobenius, hence induces the identity map on (pro-)étale sites). Now, $(\mathrm{Spa}(E)^\diamond \times_{\mathrm{Spa}\,\mathbb{F}_q} \mathrm{Spa}\,\overline{\mathbb{F}}_q)/\varphi_{\overline{\mathbb{F}}_q} = \mathrm{Spa}(\breve{E})^\diamond/\sigma$, where σ is the Frobenius of $\breve{E}$, so this is the same as σ-equivariant $\overline{\mathbb{Q}}_\ell$-local systems on $\mathrm{Spa}(\breve{E})$, which are the same as continuous $\overline{\mathbb{Q}}_\ell$-representations of W_E.
- An identification $\mathrm{Div}^d = (\mathrm{Div}^1)^d/S_d$ (the quotient is the quotient as a pro-étale sheaf).

REMARK 4.3. What is the $\underline{E}^\times$-torsor on $\mathrm{Spa}(E)^\diamond/\varphi_{E^\diamond}$ corresponding to

$$B^{\varphi=\pi} \setminus \{0\} \to [(B^{\varphi=\pi} \setminus \{0\})/\underline{E}^\times]$$

under the identification $\mathrm{Spa}(E)^\diamond/\varphi_{E^\diamond} \simeq [(B^{\varphi=\pi} \setminus 0)/\underline{E}^\times]$?

It is given by $\mathrm{Spa}(E_\infty)^\diamond \to \mathrm{Spa}(E)^\diamond/\varphi_{E^\diamond}$, where E_∞ is the Lubin-Tate extension associated to π. This can be used to verify that the identification of W_E^{ab} with $E^\times$ which will be obtained later coincides with the one of local class field theory.

We can make the isomorphism deduced from the above identifications

$$\mathrm{Spa}\, E^\diamond/\varphi_{E^\diamond} \simeq [(B_E^{\varphi=\pi} \setminus 0)/\underline{E}^\times]$$

explicit at the level of C-points, where C is a perfectoid algebraically closed field containing $\mathbb{F}_q$. Set $B = B_E(C)$. If $x \in B^{\varphi=\pi} - 0$, then $\mathrm{Div}(x)$ defines a C-point of Div^1. Conversely, an untilt corresponds to ξ a primitive element of degree 1, and we consider the divisor $\sum_{n\in\mathbb{Z}}[\varphi^n\xi]$. We will try to find $x \in B^{\varphi=\pi}$ such that $\mathrm{Div}(x) = \sum_{n\in\mathbb{Z}}[\varphi^n\xi]$. Assume $\xi \in \pi + W_{\mathcal{O}_E}(\mathfrak{m}_C)$. Then

$$\Pi^+(\xi) = \prod_{n\geq 0} \frac{\varphi^n(\xi)}{\pi}$$

converges in B and satisfies $\varphi(\Pi^+(\xi)) = \frac{\pi}{\xi}\Pi^+(\xi)$. Fargues and Fontaine then prove the existence of $\Pi^-(\xi) \in B - 0$ such that $\varphi(\Pi^-(\xi)) = \xi\Pi(\xi)$, well-defined up to multiplication by an element of $E^\times$, and set $x = \Pi(\xi) := \Pi(\xi)\Pi^+(\xi)$. The fact that $\Pi^-(\xi)$, and thus x, is only well defined up to an element of $E^\times$ is a shadow of the existence of the above $\underline{E}^\times$-torsor.

One can also write formulas using the universal cover of Lubin-Tate formal group laws over E. Then the logarithm of the formal group law shows up, as we already saw in the case $E = \mathbb{Q}_p$ in a previous lecture.

5. Geometric proof of local CFT

Let ρ be a continuous character $W_E \to \overline{\mathbb{Q}}_\ell^\times$. As discussed above, this is equivalent to a rank 1 $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{F}$ on $\mathrm{Div}^1_{\overline{\mathbb{F}}_q}$.

We want to show that $\mathcal{F}$ descends along $AJ\colon \mathrm{Div}^1_{\overline{\mathbb{F}}_q} \to \mathrm{Pic}^1_{\overline{\mathbb{F}}_q}$ to a rank 1 local system $A_{\mathcal{F}}$ on $\mathrm{Pic}^1_{\overline{\mathbb{F}}_q}$, which is equivalent to a continuous character $E^\times \to \overline{\mathbb{Q}}_\ell^\times$. Again it is enough to descend $\mathcal{F}^{(d)}$ on $\mathrm{Div}^d_{\overline{\mathbb{F}}_p}$ for $d \gg 0$. (But it is subtler to explain why : this involves Drinfeld-Scholze's lemma for diamonds.)

Therefore, it's enough to prove:

THEOREM 5.1 (Fargues). *For all $d \geq 3$, $B^{\varphi=\pi^d}_{\overline{\mathbb{F}}_q,E} - \{0\}$ is simply connected.*

REMARK 5.2. For $d = 2$, it's true in positive characteristic, but not written down in characteristic 0. It's not true for $d = 1$! It's not true if you replace $\overline{\mathbb{F}}_q$ by C, where C is an arbitrary algebraically closed field in characteristic p. Indeed, $B^{\varphi=\pi^d}_{C,E} - \{0\}$ is the d-dimensional open punctured disk over C, which has many Artin-Schreier covers.

Now we comment on the proof, and we use for the first time that $E = \mathbb{F}_q((\pi))$. For $S = \mathrm{Spa}(R, R^+)$,

$$B_{S,E} = \mathcal{O}(\mathcal{Y}_{S,E}) = \left\{\sum x_n\pi^n \colon x_n \in R, \lim_{n\to\infty} ||x_n||\rho^n = 0 \text{ for all } 0 < \rho < 1\right\}.$$

Then $B_{S,E}^{\varphi=\pi^d} = (R^{00})^d$. The isomorphism takes

$$\sum_{i=0}^{d-1}\sum_{k\in\mathbb{Z}}[x_i^{q^{-k}}]\pi^{i+kd} \leftarrow (x_0,\dots,x_{d-1})$$

So

$$B_{\overline{\mathbb{F}}_q}^{\varphi=\pi^d} = \operatorname{Spa}\overline{\mathbb{F}}_q[[x_0^{1/p^\infty},\dots,x_{d-1}^{1/p^\infty}]].$$

This is not analytic, but if we remove the non-analytic point $V(x_0,\dots,x_{d-1})$ then it is analytic. The category of étale coverings of $B_{\overline{\mathbb{F}}_q}^{\varphi=\pi^d} - \{0\}$ is the category of étale coverings of $\operatorname{Spa}\overline{\mathbb{F}}_q[[x_0^{1/p^\infty},\dots,x_{d-1}^{1/p^\infty}]]\backslash V(x_0,\dots,x_{d-1})$, which by decompletion (i.e., Elkik's theorem) and a GAGA type result is the same as the category of étale coverings of $\operatorname{Spec}\overline{\mathbb{F}}_q[[x_0,\dots,x_{d-1}]] \setminus V(x_0,\dots,x_{d-1})$ (here, we really need to work over $\overline{\mathbb{F}}_q$ and not over a perfectoid algebraically closed field C). Since $d \geq 2$, Zariski-Nagata purity allows you to extend over the puncture, and then you use that the formal power series ring is simply connected, by Hensel's lemma.

(Amusingly one can also prove simple connectedness of projective space using Zariski-Nagata purity theorem : see e.g. Serre's letter from October 25, 1959, in the Grothendieck-Serre correspondence.)

Remark 5.3. The proof in mixed characteristic is more involved, since the diamond $B_{\overline{\mathbb{F}}_q,E}^{\varphi=\pi^d} - \{0\}$ is not representable when $d > 1$...

Part 4

Day Four

CHAPTER 7

Local Langlands and Fargues' Conjecture
Speaker: Jared Weinstein

1. Local Langlands

1.1. The correspondence for $\mathrm{GL}_n(\mathbb{Q}_p)$. Recall the *Weil group* $W_{\mathbb{Q}_p} \subset \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ which is defined by the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{\mathbb{Q}_p} & \longrightarrow & W_{\mathbb{Q}_p} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_{\mathbb{Q}_p} & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & 0 \end{array}$$

(This is topologized with the discrete topology on $\mathbb{Z}$ and the usual Krull topology on $I_{\mathbb{Q}_p}$.)

The local Langlands correspondence relates

$$\underbrace{\left\{\begin{matrix}\text{irreducible smooth admissible}\\ \text{representations of } \mathrm{GL}_n(\mathbb{Q}_p)\end{matrix}\right\}}_{\mathcal{A}_n} \to \underbrace{\left\{\begin{matrix}n\text{-dimensional } \Phi\text{-semisimple}\\ W_{\mathbb{Q}_p} \xrightarrow{\text{cts}} \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)\end{matrix}\right\}}_{\mathcal{G}_n}$$

Theorem 1.1 (Harris-Taylor, Henniart '02). *There is a (unique) bijection*

$$\begin{aligned} \mathcal{A}_n &\leftrightarrow \mathcal{G}_n \\ \pi &\to \phi_\pi \\ \pi_\phi &\leftarrow \phi \end{aligned}$$

preserving L and ϵ-factors.

This bijection also satisfies local-global compatibility: if $\pi = \bigotimes_v' \pi_v$ is an algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$, and $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ satisfying for almost all unramified p': $\pi_{p'} \leftrightarrow \rho|_{W_{\mathbb{Q}_{p'}}}$, then $\pi_p \leftrightarrow \rho|_{W_{\mathbb{Q}_p}}$ as well.

Remark 1.2. For $G = \mathrm{GL}_n$, Henniart showed that there was a unique bijection preserving L and ϵ-factors. For general groups, it is still a mystery how to formulate the local Langlands correspondence in a rigid way.

Let $\mathcal{A}_n^{\text{disc}} \subset \mathcal{A}_n$ be the subset of discrete series, $\mathcal{G}_n^{\text{disc}} \subset \mathcal{G}_n$ be the subset of discrete Weil parameters. For $\mathcal{A}$, "discrete" means "essentially square-integrable". For $\mathcal{G}$, $\rho\colon W_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is "discrete" if the centralizer $S_\rho \subset \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ of ρ satisfies $S_\rho/Z(\mathrm{GL}_n)(\overline{\mathbb{Q}}_\ell)$ is finite.

REMARK 1.3. For $G = \mathrm{GL}_n$, the condition for $\rho \in \mathcal{G}_n$ to be "discrete" is equivalent to "indecomposable".

1.2. The case of GL_2**.** For GL_2, we have

$$\mathcal{A}_2 = \mathcal{A}_2^{\text{disc}} \sqcup \mathcal{A}_2^{\text{princ}} = \mathcal{A}_2^{\text{special}} \sqcup \mathcal{A}_2^{\text{sc}} \sqcup \mathcal{A}_2^{\text{princ}}.$$

Here:

- $\mathcal{A}_2^{\text{princ}}$ consists of principal series representations, which are those of the form

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \mapsto \chi_1(a)\chi_2(d).$$

  This is generically irreducible, and when it's irreducible it's called "principal series". The finite-dimensional subquotients of these are also considered "principal series".
- $\mathcal{A}_2^{\text{sp}}$ are the infinite-dimensional subquotients of these parabolically induced representations, when they are not irreducible.
- $\mathcal{A}_2^{sc}$ are those which do not appear in parabolic induction.

EXAMPLE 1.4. We explain how to construct a supercuspidal representation. It will be induced from a representation of $\mathrm{GL}_2(\mathbb{Z}_p)$, which is inflated from an irreducible cuspidal representation

$$\mathrm{GL}_2(\mathbb{F}_p) \xrightarrow{\lambda} \mathrm{GL}_{p-1}(\overline{\mathbb{Q}}_\ell).$$

Such λ are in bijection with characters $\chi\colon \mathbb{F}_{p^2}^\times \to \overline{\mathbb{Q}}_\ell^\times$ such that $\chi \neq \chi^p$, modulo the equivalence relation $\chi \sim \chi^p$. Let's write $\lambda = \lambda_\chi$.

If we induce this representation, it will not be irreducible. To pick out an irreducible constituent, extend χ to a character of $\mathbb{Q}_{p^2}^\times \to \mathbb{Q}_\ell^\times$ of conductor 1. This allows us to extend λ_χ to a representation of $\mathbb{Q}_p^\times \mathrm{GL}_2(\mathbb{Z}_p)$ (where $\mathbb{Q}_p^\times$ is the center), which you then induce to $\mathrm{GL}_2(\mathbb{Q}_p)$.

Correspondingly, we have a decomposition

$$\mathcal{G}_2 = \mathcal{G}_2^{\text{princ}} \sqcup \mathcal{G}_2^{\text{special}} \sqcup \mathcal{G}_2^{\text{cusp}}.$$

- The subset $\mathcal{G}_2^{\text{princ}}$ consists of representations of the form $\chi_1 \oplus \chi_2$.
- The subset $\mathcal{G}_2^{\text{cusp}}$ parametrizes irreducible representations, e.g. start with $\chi\colon \mathbb{Q}_{p^2}^\times \to \overline{\mathbb{Q}}_\ell^\times$ of conductor 1 such that $\chi^{p-1}|_{\mathbb{Z}_{p^2}^\times} \neq 1$, and view χ as a character of $W_{\mathbb{Q}_{p^2}}$, and then induce up to $W_{\mathbb{Q}_p}$.

- The subset $\mathcal{G}_2^{\text{special}}$ consists of reducible but not decomposable representations. For example, you could consider the tower of extensions

$$\begin{array}{c} \mathbb{Q}_p(\mu_{\ell^\infty}, p^{1/\ell^\infty}) \\ | \\ \mathbb{Q}_p(\mu_{\ell^\infty}) \\ | \\ \mathbb{Q}_p \end{array}$$

There is a two-dimensional representation of that factors through $\operatorname{Gal}(\mathbb{Q}_p(\mu_{\ell^\infty}, p^{1/\ell^\infty})/\mathbb{Q}_p)$, which sends

$$\sigma \mapsto \begin{pmatrix} 1 & t(\sigma) \\ 0 & \chi_\ell(\sigma) \end{pmatrix}$$

where $t(\sigma)$ is the isomorphism of tame inertia with $\mathbb{Z}_\ell(1)$. All representations in $\mathcal{G}_2^{\text{sp}}$ are twists of this.

1.3. General groups. In general, the correspondence takes the form of a map

$$\underbrace{\left\{\begin{matrix}\text{irreducible smooth admissible} \\ \text{representations of } G(\mathbb{Q}_p)\end{matrix}\right\}}_{\mathcal{A}_G} \to \underbrace{\left\{\begin{matrix} n\text{-dimensional } \Phi\text{-semisimple} \\ W_{\mathbb{Q}_p} \xrightarrow{\text{cts}} {}^L G(\overline{\mathbb{Q}}_\ell)\end{matrix}\right\}}_{\mathcal{G}_G}.$$

The L-group ${}^L G$ is obtained by a sort of contravariant process, involving dualizing the root datum of G.

It can happen that G_1, G_2 are two different groups such that ${}^L G_1 \simeq {}^L G_2$. This leads to interesting phenomena which are captured in Fargues' Conjecture.

For general G, the map $\mathcal{A}_G^{\text{disc}} \to \mathcal{G}_G^{\text{disc}}$ is supposed to be only finite-to-one (not a bijection). If $\phi \in \mathcal{G}_G^{\text{disc}}$, you define $S_\phi = Z_{\widehat{G}}(\phi)(\overline{\mathbb{Q}}_\ell)$, and then the fibers of $\pi \mapsto \phi_\pi$ are finite sets $\Pi_\phi(G)$ parametrized by representations of S_ϕ with prescribed restriction to $Z_{\widehat{G}}(\phi)(\overline{\mathbb{Q}}_\ell)$.

1.4. Local Jacquet-Langlands. We first focus on $G = \mathrm{GL}_n$. Isocrystals of dimension n are in bijection with $B(G)$, which is the set of "σ-conjugacy classes in $G(W(\overline{\mathbb{F}}_p)[1/p])$," i.e. the quotient of $G(W(\overline{\mathbb{F}}_p)[1/p])$ by the equivalence relation

$$g \sim \sigma(y) g y^{-1}.$$

Abbreviate $K = W(\overline{\mathbb{F}}_p)[1/p]$. This bijection takes $b \in B(G)$ to $(K^n, b\sigma)$.

For general G, $|\operatorname{Bun}_G| \simeq B(G)$. This $B(G)$ is called "Kottwitz' set". It can be equipped with a canonical topology, which we will not describe. Given $b \in B(G)$, we can define an algebraic group $J_b/\mathbb{Q}_p$ such that

$$J_b(\mathbb{Q}_p) = \{g \in G(K) \mid \sigma(g) = bgb^{-1}\}.$$

If $b = 1$, then $J_b = G$. in general, J_b is a twisted Levi of G. Exactly when b is basic, then J_b is a form of G. In fact, $J_{b,K} \simeq G_K$. (For general G, the notion of basic means that the attached character $\nu_b \colon \mathbb{D} \to G_K$ factors through $Z(G_K)$.)

Local Langlands predicts (for $G = \mathrm{GL}_n$) that there should be a canonical bijections

$$\mathcal{A}^{\mathrm{disc}}_{\mathrm{GL}_n} \leftrightarrow \mathcal{A}^{\mathrm{disc}}_{J_b}.$$

(The situation is more complicated for general G.) In general, $J_b = \mathrm{GL}_r(D)$ for $D/\mathbb{Q}_p$ a central division algebra of dimension d^2, where $n = rd$.

THEOREM 1.5 (Jacquet-Langlands, Rogawski). *There exists a unique bijection*

$$\mathcal{A}^{\mathrm{disc}}_{\mathrm{GL}_n} \xrightarrow{\sim} \mathcal{A}^{\mathrm{disc}}_{J_b},$$

denoted $\pi \mapsto \pi_b$, satisfying

$$\operatorname{Tr} \pi(g) = (-1)^{n-r} \operatorname{Tr} \pi_b(g')$$

whenever g, g' have the same irreducible (over $\mathbb{Q}_p$) characteristic polynomial.

REMARK 1.6. Although π is infinite-dimensional, Harish-Chandra made sense of $\operatorname{Tr} \pi$. This is a priori a distribution, but it is represented by a function on the locus of irreducible characteristic polynomials.

In general, there is some relationship between $\Pi_\phi(G)$ and $\Pi_\phi(J_b)$ but it is not a bijection.

2. Preparations for Fargues' Conjecture

2.1. Étale cohomology of diamonds. [Reference: Scholze's book "Étale cohomology of diamonds".]

Given a stack $\mathcal{X}$ on Perf (with the v-topology), there exists a triangulated category $D_{\text{ét}}(\mathcal{X}; \overline{\mathbb{Q}}_\ell)$ [actually, the theory at present is only developed with torsion coefficients] satisfying Grothendieck's 6-functor formalism. In particular, if $f \colon \mathcal{X} \to \mathcal{Y}$ which is nice (i.e. representable in locally spatial diamonds, and finiteness conditions) then there exist functors $f_*, f^*, f_!, f^!$ between $D_{\text{ét}}(\mathcal{X}; \overline{\mathbb{Q}}_\ell)$ and $D_{\text{ét}}(\mathcal{Y}; \overline{\mathbb{Q}}_\ell)$.

If $\mathcal{X} = X$ is a locally spatial diamond, then $D_{\text{ét}}(\mathcal{X}; \overline{\mathbb{Q}}_\ell) = D(X_{\text{ét}}; \overline{\mathbb{Q}}_\ell)$.

EXAMPLE 2.1. If $\mathcal{X} = [*/\underline{G(\mathbb{Q}_p)}]$, then $D_{\text{ét}}(\mathcal{X}; \overline{\mathbb{Q}}_\ell)$ should be the derived category of smooth representations of $G(\mathbb{Q}_p)$ on $\overline{\mathbb{Q}}_\ell$-vector spaces.

2.2. Geometric Satake. In the classical setting, with k a field, we define the affine Grassmannian as a functor on k-algebras. It is the sheafification of

$$R \mapsto \mathrm{Gr}_{\mathrm{GL}_n}(R) = \mathrm{GL}_n(R((t)))/\,\mathrm{GL}_n(R[[t]])$$

for the fpqc topology.

There is a moduli description: $\mathrm{Gr}_{\mathrm{GL}_n}$ classifies "modifications of vector bundles", i.e. pairs $\mathcal{O}^n, \mathcal{E}$ on $R[[t]]$ together with an isomorphism $\mathcal{O}^n \simeq \mathcal{E}$ away from $t = 0$.

Recall the Hecke stack

$$\begin{tikzcd} & \mathrm{Hecke} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \times X \end{tikzcd}$$

which classifies

$$(\mathcal{E}_1, \mathcal{E}_2, x, i\colon \mathcal{E}_1|_{X\setminus\{x\}} \xrightarrow{\sim} \mathcal{E}_2|_{X\setminus\{x\}}).$$

The fibers of the map Hecke $\to \mathrm{Bun}_n \times X$ are essentially affine Grassmannians.

We have an action of $G(k[[t]])$ on $\mathrm{Gr}_{\mathrm{GL}_n} = G(k((t)))/G(k[[t]])$. The space of orbits is

$$G(k[[t]])\backslash G(k((t)))/G(k[[t]]).$$

The Geometric Satake equivalence is an equivalence of $\otimes$-categories between $G(k[[t]])$-equivariant perverse $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathrm{Gr}_{\mathrm{GL}_n}$ and $\mathrm{Rep}(\mathrm{GL}_n)$.

Example 2.2. The substack $\mathrm{Gr}^{\mathrm{elem},d}_{\mathrm{GL}_n} \subset \mathrm{Gr}_{\mathrm{GL}_n}$ classifies extensions

$$\mathcal{O}^n \to \mathcal{E} \to i_* W$$

over $R[[t]]$, where $i\colon \mathrm{Spec}\, R \to \mathrm{Spec}\, R[[t]]$ and W is locally free of rank d over R.

Then $\mathrm{Gr}^{\mathrm{elem},d}_{\mathrm{GL}_n} \simeq \mathrm{Gr}(d,n)$ (the usual Grassmannian). We have $j_d\colon \mathrm{Gr}(d,n) \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}$. Under the Satake equivalence,

$$j_{d*}\overline{\mathbb{Q}}_\ell[\dim \mathrm{Gr}(d,n)] \leftrightarrow \wedge^d \mathrm{Std}.$$

2.3. The B_{dR}-affine Grassmannian. In the p-adic setting, we can define a sheaf on $\mathrm{Perf}_{\mathbb{Q}_p}$:

$$\mathrm{Gr}^{B_{\mathrm{dR}}}_{\mathrm{GL}_n}(S) = \{\text{modifications } \mathcal{O}^n \xrightarrow{\sim} \mathcal{E} \text{ on } \mathcal{X}_{S^\flat} \setminus S\}.$$

If $S = \mathrm{Spa}(R, R^+)$, then $B^+_{\mathrm{dR}}(R) = \widehat{\mathcal{O}}_{X_{R^\flat}, R}$ which is the completion of $A_{\mathrm{inf}}(R^{+\flat})[1/p]$ at ξ, and $B_{\mathrm{dR}}(R) = B^+_{\mathrm{dR}}(R)[1/\xi]$. Then $\mathrm{Gr}^{B_{\mathrm{dR}}}_{\mathrm{GL}_n}$ is the sheafification of

$$S = \mathrm{Spa}(R, R^+) \mapsto \mathrm{GL}_n(B_{\mathrm{dR}}(R))/\mathrm{GL}_n(B^+_{\mathrm{dR}}(R)).$$

Note that $\mathrm{Gr}^{B_{\mathrm{dR}}}_{\mathrm{GL}_n}(R, R^+)$ doesn't depend on the choice of R^+.

Theorem 2.3 (Fargues, Scholze). *There exist a Satake equivalence for $\mathrm{Gr}^{B_{\mathrm{dR}}}_G$, relating perverse sheaves on $\mathrm{Gr}^{B_{\mathrm{dR}}}_G$ and $\mathrm{Rep}(\widehat{G})$.*

3. The Conjecture

Now in the setting of the Fargues-Fontaine curve, we have a correspondence

$$\begin{tikzcd} & \mathrm{Hecke} \arrow[dl, "h^\leftarrow"'] \arrow[dr, "h^\rightarrow"] & \\ \mathrm{Bun}_{\mathrm{GL}_n} & & \mathrm{Bun}_{\mathrm{GL}_n} \times \mathrm{Div}^1_{\mathbb{F}_q} \end{tikzcd}$$

The Satake equivalence $\mathrm{Rep}^{\mathrm{alg}}_{\mathrm{GL}_n} \xrightarrow{\sim} \mathrm{Perv}(\mathrm{Gr}^{B_{\mathrm{dR}}}_{\mathrm{GL}_n})$ sends $V \mapsto \mathcal{S}_V$.

Conjecture 3.1. *Given $\phi \in \mathcal{G}^{\mathrm{disc}}_{\mathrm{GL}_n}$, there exists $\mathcal{F}_\phi \in D_{\acute{e}t}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ with the following properties:*

(1) *For each* $b \in B(G)$*, we have an open substack*
$$[*/\underline{J_b(\mathbb{Q}_p)}] \simeq \mathrm{Bun}^b_{\mathrm{GL}_n} \hookrightarrow \mathrm{Bun}_{\mathrm{GL}_n}.$$
Then $x_b^* \mathcal{F}_\phi$ *is identified with* π_b*, where* $\pi = \pi_\phi$*.*

(2) *(Eigensheaf property) For all* $V \in \mathrm{Rep}(\mathrm{GL}_n)$*,*
$$h^{\rightarrow}_!(h^{\leftarrow *}\mathcal{F}_\phi \otimes \mathcal{S}_V) \simeq \mathcal{F}_\phi \boxtimes (V \circ \phi)$$

(3) *(Cuspidality) If* ϕ *is irreducible, then* $\mathcal{F}_\phi$ *is supported on*
$$\mathrm{Bun}^{ss}_{\mathrm{GL}_n} = \coprod_{b \in B(G)_{\mathrm{basic}}} [*/\underline{J_b(\mathbb{Q}_p)}].$$

CHAPTER 8

Drinfeld's proof of the global Langlands correspondence for $\mathrm{GL}(2)$
Speaker: Sophie Morel

1. Overview

Let $X/\mathbb{F}_q$ be a smooth projective geometrically connected curve, $F = \mathbb{F}_q(X)$. We assume $g(X) \geq 2$ since we want to consider irreducible 2-dimensional representations of $\pi_1(X)$.

1.1. First goal. Let $n \geq 2$; often we will take $n = 2$. Given an irreducible representation $\rho\colon \pi_1(X) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, we want to construct a non-zero function

$$f_\rho \in C_c(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/\mathrm{GL}_n(\mathcal{O}), \overline{\mathbb{Q}}_\ell)$$

which is a *Hecke eigenfunction* with respect to ρ, i.e. for all $x \in |X|$, and all $T_v \in \mathcal{H}_{\mathrm{sph},x} = C_c(G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x); \overline{\mathbb{Q}}_\ell)$, we have

$$f_\rho * T_x = \sigma_x(T_X) f_\rho$$

where $\sigma_x\colon \mathcal{H}_{\mathrm{sph},x} \to \overline{\mathbb{Q}}_\ell$ corresponds to $\rho(\mathrm{Frob}_x)$ under the Satake isomorphism.

Drinfeld constructed f_ρ as the trace of Frobenius associated to a sheaf. The idea is that

$$\mathrm{Bun}_n(\mathbb{F}_q) = \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathcal{O}),$$

and then f_ρ could come from an ℓ-adic sheaf on Bun_n.

1.2. Function-sheaf dictionary. Let $Y \to \operatorname{Spec} \mathbb{F}_q$ be a nice stack. For $K \in D(Y) := D^b_c(Y; \overline{\mathbb{Q}}_\ell)$ we get a function

$$f_K\colon Y(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$$

sending $y \mapsto \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{Tr}(\mathrm{Frob}_q, \mathcal{H}^i(K_y))$.

Note that we can interpret a representation $\rho\colon \pi_1(X) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ as a rank n $\overline{\mathbb{Q}}_\ell$-local system E on X.

We wish to construct a sheaf Aut_E on Bun_n such that $\mathcal{F}_{\mathrm{Aut}_E}$ is a non-zero eigenform with respect to E. This is an unnatural condition to put on a sheaf, so we replace it with the concept of an *eigensheaf*.

1.3. Eigensheaves.

1.3.1. *Weak condition.* Let Hecke^1 be the stack parametrizing elementary modifications $\mathcal{E} \hookrightarrow \mathcal{E}'$ of rank length at x.

$$\begin{tikzcd} & \mathrm{Hecke}^1 \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow}"] \arrow[dd, "\mathrm{supp}"] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \\ & X & \end{tikzcd}$$

Using this we have a functor $H^1 \colon D(\mathrm{Bun}_n) \to D(X \times \mathrm{Bun}_n)$ sending

$$K \mapsto (\mathrm{supp} \times h^{\rightarrow})_!(h^{\leftarrow *}K \otimes \overline{\mathbb{Q}}_\ell(\frac{n-1}{2})[n-1])$$

(All functors are derived.)

Definition 1.1. $K \in D(\mathrm{Bun}_n)$ is a *weak Hecke eigensheaf* with respect to E if $K \neq 0$, and if there exists an isomorphism $H^1(K) \simeq E \boxtimes K$ such that $(H^1)^{\boxtimes 2}(K) \simeq E \boxtimes E \boxtimes K$ is S_2-equivariant.

1.3.2. *Less weak version.* The weak condition actually implies the following stronger condition. We consider all stacks Hecke^d parametrizing elementary modifications of degree d at x. There is a similar diagram

$$\begin{tikzcd} & \mathrm{Hecke}^d \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow}"] \arrow[dd, "\mathrm{supp}"] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \\ & X & \end{tikzcd}$$

We can then define $H^d \colon D(\mathrm{Bun}_n) \to D(X \times \mathrm{Bun}_n)$ sending

$$K \mapsto (\mathrm{supp} \times h^{\rightarrow})_!(h^{\leftarrow *}K \otimes \overline{\mathbb{Q}}_\ell(\frac{d(n-d)}{2})[d(n-d)])$$

Definition 1.2. We say K is a *Hecke eigensheaf* if $K \neq 0$ and if for all $d \in \{1, \dots, n\}$,

$$H^d(K) \simeq \wedge^d E \boxtimes K.$$

Remark 1.3. Actually, the weak version implies the "less weak version". Actually (1) even implies the "strong eigensheaf property" involving any stratum of Hecke_{X^I}.

1.4. Updated goal. Let k be a field, e.g. $k = \mathbb{F}_q$. Let X be a curve over k. Given E a local system on X that is geometrically irreducible, find $\mathrm{Aut}_E \in D(\mathrm{Bun}_n)$ that is a Hecke eigensheaf with respect to E.

We will moreover want $\mathrm{Aut}_E \in \mathrm{Perv}(\mathrm{Bun}_n)$ that is a Hecke eigensheaf with respect to E and irreducible on every Bun_n^d. This guarantees that Aut_E is non-zero, since there is a dense open subset where it is a non-zero local system. (A priori one would be worried about cancellation.)

Having said this, we will give an argument that only works in positive characteristic.

2. Idea of the construction

The idea is to use the Whittaker model. What is this?

For simplicity we will discuss the case of $n = 2$ only. Let $B \supset N$ be the usual Borel with its unipotent radical.

Fix a non-trivial character $\Psi\colon \mathbb{A}/F \to \overline{\mathbb{Q}}_\ell^\times$.

The *Whittaker space* is the space of functions

$$\varphi\colon (N(\mathbb{A}), \Psi)\backslash \mathrm{GL}_2(\mathbb{A})/\mathrm{GL}_2(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell$$

where the meaning of $(N(\mathbb{A}), \Psi)\backslash$ is that

$$\varphi(ux) = \Psi(u)\varphi(x) \text{ for all } u \in N(\mathbb{A}).$$

We also assume that everything has trivial central character, and that appropriate growth conditions are imposed. Then the previous space coincides with

$$\{f\colon B(F)\backslash \mathrm{GL}_2(\mathbb{A})/\mathrm{GL}_2(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell \text{ cuspidal}\}.$$

The bijection is given by

$$\left(\varphi(x) :=\mapsto \int_{N(F)\backslash N(\mathbb{A})} f(ux)\psi(u^{-1})\, du\right) \leftarrow f$$

and

$$\varphi \mapsto \left(f(x) :=\mapsto \sum_{a\in F^\times} \varphi\left(\begin{pmatrix} a & \\ & 1\end{pmatrix} x\right)\right)$$

This bijection is Hecke-equivariant, and we also know that the the Whittaker model has multiplicity one.

The *Casselman-Shalika formula* gives an explicit description of a Whittaker function with a given Hecke eigenvalue.

3. Geometric version of $B(F)\backslash \mathrm{GL}_2(\mathbb{A})/\mathrm{GL}_2(\mathcal{O})$

We will construct a stack Bun_2' with a proper map $\mathrm{Bun}_2' \to \mathrm{Bun}_2$. Then

$$\mathrm{Bun}_2' = \{\mathcal{E} \in \mathrm{Bun}_2, s\colon \Omega^1_X \hookrightarrow \mathcal{E}\}.$$

In this and all moduli descriptions that follow, all constructions are demanded to be flat over S; in particular, the cokernel of s is flat over S.

4. The Laumon sheaf

Definition 4.1. Let Coh_0^d be the moduli space of torsion $\mathcal{O}_X$-modules (the subscript 0 indicates the generic rank) of length d (flat over S). Define $\mathrm{Coh}_0 = \coprod_{d\in\mathbb{Z}} \mathrm{Coh}_0^d$.

There is a map $\mathrm{Hecke}^d \to \mathrm{Coh}_0^d$ sending $(\mathcal{E} \hookrightarrow \mathcal{E}') \mapsto \mathcal{E}'/\mathcal{E}$.

There is a map $X^d/S_d = X^{(d)} \to \mathrm{Coh}_0^d$ sending $D \mapsto \mathcal{O}_{X,D}$. There is also a (non-representable) morphism $\mathrm{Coh}_0^d \to X^{(d)}$ which sends $\bigoplus_i \mathcal{O}_{X,D_i} \mapsto \sum D_i$. The composition

$$X^{(d)} \to \mathrm{Coh}_0^d \to X^{(d)}$$

is the identity map.

Example 4.2. Let $X = \mathbb{A}^1$. Then $\mathrm{Coh}_0^d(A)$ is the set of torsion $A[t]$-modules of length d, flat over A. Viewing an $A[t]$-module as an A-module with endomorphism, we can re-interpret $\mathrm{Coh}_0^d(A)$ as the set of locally free A-modules of rank d plus $\rho \in \mathrm{End}_A(V) = [\mathfrak{gl}_d/\mathrm{GL}_d]$.

Let's unravel the maps

$$X^{(d)}(A) \xrightarrow{\pi} \mathrm{Coh}_0^d(A) \xrightarrow{\det} X^{(d)}(A).$$

The second map takes a matrix to the zeros of its characteristic polynomial (viewed as a divisor on A^1). The first map can be described concretely away from the "large diagonal" Δ (the union of all loci where coordinates coincide): it takes $a_1, \dots, a_d \in A^d$ to the matrix $\mathrm{diag}(a_1, \dots, a_d)$. The map is a torsor under $\mathbb{G}_m^d$ away from the diagonal, so we see the map is smooth of relative dimension d. (In fact it is true that π is smooth of relative dimension d everywhere.)

Let E be a local system on X. We have $\mathrm{sym}\colon X^d \to X^{(d)}$.Then we can form $E^{\boxtimes d}$ on X^d. Then we make $E^{(d)} := (\mathrm{sym}_* E^{\boxtimes d})^{S_d}$. This enjoys the following properties:

- $E^{(d)}[-d]$ is perverse (since it's a summand of the pushforward of a perverse sheaf by a finite map), irreducible if E is. The stalk over a divisor D is

$$\bigotimes_{x\in|X|} \mathrm{Sym}^{\deg x(D)} E_x.$$

- $E^{(d)}$ is a local system on $X^{(d)} - \Delta$.
- $E^{(d)} = \pi^* \det^* E^{(d)}$, so in particular $E^{(d)}$ descends through π.

Definition 4.3. Let $\mathrm{Coh}_0^{d,\mathrm{rss}} := \det^{-1}(X^{(d)} - \Delta)$. We define $\mathscr{L}_E^{d\circ} := \det^* E^{(d)}|_{\mathrm{Coh}_0^{d,\mathrm{rss}}}[d]$. We define the *Laumon sheaf* $\mathscr{L}_E^d$ be the intermediate extension to Coh_0^d, and $\mathscr{L}_E := \bigoplus_{d\in\mathbb{N}} \mathscr{L}_E^d$.

5. Whittaker sheaf

Fix $\psi\colon \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$ a non-trivial character. (This will be the source of the Ψ.) Then we get an Artin-Schreier sheaf $\mathcal{J}_\psi$ on $\mathbb{A}^1$. (This is built using the Lang isogeny $L\colon \mathbb{A}^1 \to \mathbb{A}^1$.)

We will define the "Whittaker functional" sheaf. There are actually several constructions of the Whittaker sheaf, and right now we are presenting the third construction.

Definition 5.1. We define the moduli space $\mathcal{Q}$ whose S-points are

$$\{\mathcal{E} \in \mathrm{Bun}_2(S) \colon s_1 \colon \Omega^1_X \hookrightarrow \mathcal{E}, s_2 \colon \Omega^1_x \hookrightarrow \wedge^2\mathcal{E}\}$$

such that such that (s_1, s_2) define a complete flag of subbundles genericallly.

This has a stratification

$$\overline{\mathcal{Q}} = \coprod_{d \in \mathbb{Z}} \overline{\mathcal{Q}}^d$$

where $\overline{\mathcal{Q}}^d$ parametrizes $\mathcal{E}$ of degree d.

There is an open substack $j \colon \mathcal{Q} \hookrightarrow \overline{\mathcal{Q}}^0$ where $\operatorname{coker} s_i$ is torsion-free. There is a map

$$\mathrm{ev} \colon \mathcal{Q} \to \mathbb{A}^1$$

by sending $\underline{\mathrm{Ext}}(\mathcal{O}_X, \Omega^1_X) \mapsto \mathrm{Ext}^1(\mathcal{O}_X, \Omega^1_X)$.

Definition 5.2. We define another stack Z^d as the fibered product

$$\begin{tikzcd} Z^d \arrow[r] \arrow[d] & \mathrm{Hecke}^d \arrow[d, "h^{\leftarrow}"] \\ \overline{\mathcal{Q}}^0 \arrow[r, "\mathcal{E}"] & \mathrm{Bun}_2 \end{tikzcd}$$

This parametrizes $(\mathcal{E}, s_1, s_2, \beta \colon \mathcal{E} \hookrightarrow \mathcal{E}')$.

There is a map $Z^d \to \overline{\mathcal{Q}}^d$ sending by

$$(\mathcal{E}, s_1, s_2, \beta \colon \mathcal{E} \hookrightarrow \mathcal{E}') \mapsto (\mathcal{E}', \beta \circ s_i).$$

We have a diagram

$$\begin{tikzcd} & Z^d \arrow[dl, "(h^{\rightarrow})'"'] \arrow[dr, "p'"] \arrow[dd, "q'"] & & \\ \overline{\mathcal{Q}}^d & & \overline{\mathcal{Q}}^0 & \\ & \mathrm{Hecke}^d \arrow[r, "\pi"] & \mathrm{Coh}^d_0 & \end{tikzcd}$$

Definition 5.3. We define

$$\Psi^0 = j_! \mathrm{ev}^*(\mathcal{J}_\psi) \otimes \overline{\mathbb{Q}}_\ell(\ldots)[\ldots] \in D(\overline{\mathcal{Q}}^0).$$

(Shifts and twists are uniquely determined by asking Ψ^0 to have weight 0.) We define

$$\mathscr{W}^d_E = h^{\rightarrow'}_! ((p')^*(\Psi^0) \otimes (\pi \circ q')^*(\mathscr{L}_E))(...)[...] \in D(\overline{\mathcal{Q}}^d).$$

The point is that $\overline{\mathcal{Q}}^d(\mathbb{F}_q) \hookrightarrow N(F)\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{G}_2(\mathcal{O})$, and $f_{W_E^d}$ is the restriction of the Whittaker functional with eigenvalue E.

We have a map $v' \colon \overline{\mathcal{Q}} \mapsto \mathrm{Bun}_2'$ sending

$$(\mathcal{E}, s_1 \colon \Omega^1 \hookrightarrow \mathcal{E}, s_2 \colon \Omega^1 \hookrightarrow \wedge^2 \mathcal{E}) \mapsto (\mathcal{E}, s_1 \colon \Omega^1 \hookrightarrow \mathcal{E})$$

Definition 5.4. We define $\mathrm{Aut}_E' = (v')_!(\bigoplus_{d\in\mathbb{Z}} \mathscr{W}_E^d) \in D(\mathrm{Bun}_2')$.

Fact 5.5. Aut_E' *is a Hecke eigensheaf.*

So we have a Hecke eigensheaf on "$B(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)/\mathrm{GL}_2(\mathcal{O})$".

6. Second construction

We are going to make a second construction of the automorphic sheaf, which gives a description more amenable to descent.

Lemma 6.1. *Let* $\mathscr{L} \in \mathrm{Pic}_X(\overline{k})$ *be ample enough, e.g. we'll be interested in* $\mathscr{L} = \Omega^3 = (\Omega^1)^{\otimes 3}$.

Then there exists a constant $c(g)$ *such that for all* $d \geq c(g)$, *and all* $\mathcal{E} \in \mathrm{Bun}_n^d(\overline{k})$,

$$\mathrm{Hom}(\mathcal{O}_X, \mathscr{L}) \neq 0 \implies \mathscr{E} \text{ is very unstable}$$

i.e. $\mathcal{E} \simeq \mathcal{E}_1 \oplus \mathcal{E}_2$ *such that* $\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2) = 0$.

Take $\mathscr{L} = \Omega^3$. Then for $0 \leq i \leq n = 2$, we have an open immersion

$$\mathcal{C}_i \hookrightarrow \mathrm{Coh}_i := \{\text{coherent } \mathcal{O}_X\text{-modules of generic rank } i\}$$

and $\mathcal{C}_i$ is the substack where $\deg M \geq c(g) + i(i-1)(g-1)$ and $\mathrm{Hom}(M, \mathscr{L}) = 0$. This implies $\mathrm{Hom}(M, \Omega^i) = 0$ for $0 \leq i \leq 2$, and then by Serre duality that $\mathrm{Ext}^1(\Omega^{i-1}, M) = 0$.

Then define $\mathcal{E}_i = \{M \in \mathcal{C}_i, s \colon \Omega^{i-1} \to M\}$. The map $(M, s) \mapsto M$ makes $\mathcal{E}_i$ a vector bundle over $\mathcal{C}_i$, thanks to the property $\mathrm{Ext}^1(\Omega^{i-1}, M) = 0$ of $M \in \mathcal{C}_i$. There is an open substack $\mathcal{E}_i^\circ \subset \mathcal{E}_i$ where s is injective. The dual bundle is $\mathcal{E}_i^\vee$.

$$\begin{tikzcd} \mathcal{E}_i \arrow[dr, "\rho_i"] & & \mathcal{E}_i^\vee \arrow[dl, "\rho_i^\vee"] \\ & \mathcal{C}_i & \end{tikzcd}$$

This parametrize not subbundles but extensions, by a geometric form of Serre duality: $\mathcal{E}_i^\vee$ parametrizes $0 \to \Omega^i \to M' \to M \to 0$.

Denote the map $\rho_i^\vee \colon \mathcal{E}_i^\vee \to \mathcal{C}_i$. There is an open substack $\mathcal{E}_i^{\vee 0} \hookrightarrow \mathcal{E}_i^\vee$ defined by the extra condition that $M' \in \mathcal{C}_{i+1}$.

We have $\mathcal{E}_i^\circ \simeq \mathcal{E}_{i-1}^{\vee\circ}$, by using (M, s) to make

$$0 \to \Omega^{i-1} \xrightarrow{s} M \to \mathrm{coker}(s) \to 0.$$

Also $\mathcal{E}_2^\circ$ is an open substack of Coh_2', the analogue of Bun_2' but with coherent sheaves instead of vector bundles: $\mathrm{Coh}_2' = \{(M \in \mathrm{Coh}_2, s\colon \Omega^1 \hookrightarrow M)\}$. The square

$$\begin{tikzcd} \mathcal{E}_2^\circ \arrow[r, hook] \arrow[d] & \mathrm{Coh}_2' \arrow[d] \\ \mathcal{C}_2 \arrow[r, hook] & \mathrm{Coh}_2 \end{tikzcd}$$

is Cartesian.

There is the following fundamental diagram:

$$\begin{tikzcd} \mathcal{E}_2 \arrow[d, "\rho_2"] & \mathcal{E}_2^\circ \simeq \mathcal{E}_1^{\vee\circ} \arrow[l, hook', "j_2"'] \arrow[r, hook] & \mathcal{E}_1^\vee \arrow[dr, "\rho_1^\vee"'] & & \mathcal{E}_1 \arrow[dl, "\rho_1"] & \mathcal{E}_1^\circ \simeq \mathcal{E}_0^{\circ\vee} \arrow[l, hook', "j_1"'] \arrow[r, hook] & \mathcal{E}_0^\vee \arrow[dr, "\rho_0^\vee"] & \\ \mathcal{C}_2 & & & \mathcal{C}_1 & & & & \mathcal{C}_0 \subset \mathrm{Coh}_0 \end{tikzcd}$$

We define

$$\begin{aligned} \mathcal{F}_{E,1} &:= (\rho_0^{\vee *}\mathscr{L}_E)|_{\mathcal{E}_0^{\circ\vee} \simeq \mathcal{E}_1^\circ} \\ \mathcal{F}_{E,2} &:= \mathrm{Four}(j_{1!}\mathcal{F}_{E,1})|_{\mathcal{E}_1^{\vee 0} \simeq \mathcal{E}_2^0}. \end{aligned}$$

Fact 6.2. *We have*

$$\mathcal{F}_{E,2}|_{\mathcal{E}_2^\circ \cap \mathrm{Bun}_2'} \simeq \mathrm{Aut}_E' |_{\mathcal{E}_2^\circ \cap \mathrm{Bun}_2'}.$$

Key point: $\mathcal{F}_{E,1}$ is "clean", i.e. $j_{1!}\mathcal{F}_{E,1} \xrightarrow{\sim} j_{1*}\mathcal{F}_{E,1}$.

Corollary 6.3. *The sheaf* $j_{1!}\mathcal{F}_{E,1} = j_{1!*}\mathcal{F}_{E,1}$ *is perverse.*

Since the ℓ-adic Fourier transform preserves perversity, we alsog et that $\mathcal{F}_{E,2}$ is perverse, and is irreducible if E is (which we are assuming).

The clean-ness follows from the Vanishing Theorem proved by Gaitsgory.

The advantage of perversity is that it's easier to descend perverse sheaves.

We have $\mathcal{F}_{E,2}$ on $\mathcal{E}_2^\circ$.

$$\begin{tikzcd} \mathcal{E}_2^\circ \arrow[r, hook] \arrow[d] & \mathrm{Coh}_2' \arrow[d] \\ \mathcal{C}_2 \arrow[r] & \mathrm{Coh}_2 \end{tikzcd}$$

Observation: if $d \gg 0$,

$$\begin{tikzcd} (\mathcal{E}_2^\circ)^d \arrow[r, hook] \arrow[d, "\pi"] & (\mathcal{E}_2)^d \arrow[dl] \\ \mathcal{C}_2^d & \end{tikzcd}$$

It's easy to show that $\mathcal{F}_{E,2}$ is $\mathbb{G}_m$-equivariant, so it descends to $\mathbb{P}(\mathcal{E}_2)$ over $\mathcal{C}_2^d$. Then we want to descend through a projective bundle, which will be true if it's constant enough on fibers.

PROPOSITION 6.4. *Let $\rho\colon \mathcal{V} \to S$ be a geometric vector bundle of rank r. Let $K \in \mathrm{Perv}(\mathbb{P}(V))$ be irreducible. The following are equivalent:*

(1) K descends to something on S,
(2) The Euler characteristic $\chi_K\colon \mathcal{V}(\bar{k}) \to \mathbb{Z}$ is constant along the fibers of $\mathbb{P}(V) \to S$.

We know K is locally constant on some dense open. We need to show its restriction to this open descends. To get a big enough open, we use the following Lemma: a perverse sheaf on a smooth scheme with constant Euler characteristic is a local system.

To get the second condition, we use a result of Deligne. The result of Deligne will show that $\chi_{\mathcal{F}_{E,2}}$ does not depend on the rank 2 local system E. This reduces to proving the constancy of Euler characteristics for a favorably chosen E.

CHAPTER 9

"Where does the conjecture come from? What happened in Orsay, Trieste and Berkeley?" Part I Speaker: Laurent Fargues

Warning: The conjecture does not come from saying "let's copy-paste the Geometric Langlands program to the setting of the Curve".

Rather, it comes from thinking about and studying the following objects:

- p-adic period morphisms (de Rham and Hodge-Tate),
- p-adic geometry of Shimura varieties and Rapoport-Zink spaces,
- cohomology of these and *Igusa varieties*,
- the work of Vogan, Kottwitz, and Kaletha on the Local Langlands correspondence.

1. Starting point for the Curve

1.1. Finite flat group schemes. It began (for me) by studying HN filtrations of finite flat group schemes.

Let $K/\mathbb{Q}_p$ be a complete field, with a not necessarily discrete valuation. (I was studying p-adic Hodge theory over algebraically closed fields C, but not with an eye towards Galois representations.)

Let $G/\mathcal{O}_K$ be a finite flat group scheme, of order a power of p. I defined the *height* of G to be $\log_p |G|$, and the *degree* of G to be $\sum_i v(a_i)$ where $\omega_G = \bigoplus_i \mathcal{O}_K/a_i\mathcal{O}_K$ is the Dieudonné module. Here the valuation is normalized so that $v(p) = 1$.

Then we define $\mu(G) = \frac{\deg G}{\mathrm{ht}(G)} \in [0,1]$. This lets us develop a theory of HN filtrations.

1.2. p-divisible groups. The next step occurred in my paper "Théorie de la réduction pour les groups p-divisibles". Let $H/\mathcal{O}_K$ be a p-divisible group (e.g. $H = A[p^\infty]$ for an abelian scheme $A/\mathcal{O}_K$).

For $n \geq 1$, you can consider $\mathrm{HN}(H[p^n])$ as a function $[0, n\mathrm{ht}(H)] \to [0, n\dim H]$.

Definition 1.1. We define the *renormalized Harder-Narasimhan polygon* to be the concave function $[0, \mathrm{ht}(H)] \to [0, \dim H]$ defined by

$$x \mapsto \lim_{n\to\infty} \frac{1}{n}\mathrm{HN}(H[p^n])(nx).$$

Remark 1.2. If P_n is $\mathrm{HN}(H[p^n])$, then $P_{n+m} \leq P_n * P_m$ where $*$ is "concatenation". This is what implies the convergence of the limit.

This definition led to the question: is $\mathrm{HN}(H)$ a polygon?

1.3. The Hodge-Tate period map. The solution to this question came from trying to linearize the non-linear objects that are finite flat group schemes. This meant using Hodge-Tate periods of finite flat group schemes.

Recall: we are assuming $K = \widehat{\overline{K}} = C$. (We can always reduce to this case, because the HN filtration satisfies Galois descent.)

Let $H/\mathcal{O}_C$ be a p-divisible group. Then there is a *Hodge-Tate period morphism*

$$\alpha_H \colon V_p(H) \to \omega_{H^D}[1/p]$$

sending

$$x \mapsto (x^D)^* \frac{dT}{T}.$$

Here we are viewing $x \in T_p(H)$ as $x \colon \mathbb{Q}_p/\mathbb{Z}_p \to H$, so $x^D \colon H^D \to \widehat{\mathbb{G}}_m$.

The linearization

$$\alpha_H \otimes 1 \colon V_p(H) \otimes_{\mathbb{Q}_p} C \to \omega_{H^D}[1/p]$$

is surjective. (This is the right half of the Hodge-Tate exact sequence.)

We can axiomatize this situation to triples $X = (V, W, \beta)$ with V a finite-dimensional $\mathbb{Q}_p$-vector space, W a finite-dimensional C-vector space, and $\beta \colon V_C \twoheadrightarrow W$. Set

$$\deg X = \dim_C(\ker \beta)$$
$$\text{rank } X = \dim_{\mathbb{Q}_p} V.$$

We then define the slope $\mu(X) := \frac{\deg X}{\text{rank } X}$.

The function μ equips the category of (V, W, β) with the structure of an HN category.

THEOREM 1.3. *The renormalized HN polygon satisfies*

$$\mathrm{HN}(H) = \mathrm{HN}(V_p(H), \omega_{H^D}[1/p], \alpha_H \otimes 1).$$

In particular, $\mathrm{HN}(H)$ *is a polygon.*

1.4. Banach-Colmez spaces. Next I wanted to prove:

THEOREM 1.4. *The renormalized HN polygon satisfies*

$$\mathrm{HN}(H) \leq \mathrm{Newt}(H_{k_C})$$

for k_C *the residue field of* $\mathcal{O}_C$.

REMARK 1.5. The motivation was for studying the stratification of Shimura varieties. This is "easy" in the situation of $H/\mathcal{O}_K$ where $[K : \mathbb{Q}_p] < \infty$, using p-adic Hodge theory. This gives a relation between the HN stratification and Newton stratification of a Shimura variety.

EXAMPLE 1.6. The theorem implies that if H_{k_c} is isoclinic, then $\mathrm{HN}(H)$ is a line.

That led me to study Banach-Colmez spaces.

Recall that Banach-Colmez spaces are functors from "nice" C-algebras (nowadays we recognize "nice" as a particular case of "perfectoid") to $\mathbb{Q}_p$-Banach spaces of the form:

(extension of a C-vector space by a $\mathbb{Q}_p$-vector space)/sub $\mathbb{Q}_p$-vector space.

Now we come back to: there exists an exact sequence of Banach-Colmez spaces

$$0 \to V_p(H) \to (D \otimes B^+_{\mathrm{cris}})^{\varphi=p} \to \mathrm{Lie}\, H[1/p] \to 0$$

where D is the covariant Dieudonné module of H_{k_C}. This was later interpreted in terms of modifications of vector bundles on the Curve.

Banach-Colmez spaces form an abelian category with two additive functions, describing the "C-dimension" and the "$\mathbb{Q}_p$-dimension". (Colmez proved that these functions don't depend on the presentation.)

This was motivation to introduce a notion of HN filtration on Banach-Colmez spaces, with the two functions playing the roles of deg and rank. This implies the desired theorem, by realizing the HN polygon as a Newton polygon.

1.5. Geometrization of Banach-Colmez spaces. Later we tried to geometrize the Banach-Colmez spaces. This question was asked by Fontaine in his article for Kato's 50th birthday. It gave rise to §4 of our article on formal $\mathbb{Q}_p$-vector spaces, where we geometrize $\varprojlim_p H$.

We tried to classify Banach-Colmez spaces. We could construct a fully faithful functor

$$\{\text{some BC spaces}\} \hookrightarrow \left\{\begin{matrix} B_e\text{-modules free of finite type} \\ B^+_{\mathrm{dR}}\text{-modules free of finite type} \\ + \text{ gluing data} \end{matrix}\right\}$$

where $B_e = B^{\phi=1}_{\mathrm{cris}}$. The right hand side looked like Beauville-Laszlo gluing.

Part 5

Day Five

CHAPTER 10

"Where does the conjecture come from? What happened in Orsay, Trieste and Berkeley?" Part II
Speaker: Laurent Fargues

1. Last episode

For $H/\mathcal{O}_C$ a p-divisible group, we wanted to prove an inequality for the renormalized HN polygon of H:

$$\mathrm{HN}(H) \leq \mathrm{Newt}(H_{k_C}).$$

For this we use an exact sequence of Banach-Colmez spaces

$$0 \to V_p(H) \to (D \otimes B^+_{\mathrm{cris}})^{\varphi=1} \to \mathrm{Lie}\, H[1/p] \to 0$$

(where D is the covariant Dieudonné module of H_k) plus the fact that BC spaces have HN filtrations. These facts easily imply the result.

2. The discovery of the Curve

I ended the previous talk by explaining that we constructed a fully faithful functor

$$\{\text{some BC spaces}\} \hookrightarrow \left\{ \begin{array}{c} B_e\text{-modules free of finite type} \\ B^+_{\mathrm{dR}}\text{-modules free of finite type} \\ + \text{ gluing data} \end{array} \right\}.$$

The right hand side looked like a Beauville-Laszlo description of vector bundles.

In the meantime, Berger proved that B_e is Bezout, using results of Kedlaya. Fontaine went further and proved that B_e is a PID. Using this, he gave a short proof of "weakly admissible implies admissible".

This led to the discovery of the curve in Trieste. The first construction was as a "gluing" of Spec B_e and Spec B^+_{dR}. Later, we realized that one can define it as $\mathrm{Proj}\, P$, where P was this graded algebra of p-adic periods.

Already in Trieste I conjectured that any vector bundle on X is $\bigoplus_i \mathcal{O}(\lambda_i)$, by analogy with results of Kedlaya and Grothendieck's classification of vector bundles on $\mathbb{P}^1$. So I began to study the curve, and vector bundles on it.

3. The structure of the Curve

We tried to understand the set of closed points $|X|$. This led us to introduce the notion of "primitive elements" in A_{inf}.

Recall: for $F/\mathbb{F}_p$ a perfectoid field, $A_{\text{inf}} = W(\mathcal{O}_F)$. An element

$$\xi = \sum_{n \geq 0} [x_n] p^n \in A_{\text{inf}}$$

is called *primitive of degree* $d > 0$ if $x_0 \neq 0$, $x_1, \ldots, x_{d-1} \in \mathfrak{m}_F$, and $x_d \in \mathcal{O}_F^\times$. This notion comes from Weierstrass factorization theory.

We defined Y to be the set of irreducible primitive elements in A_{inf} up to multiplication by $A_{\text{inf}}^\times$. There is a natural embedding $Y \hookrightarrow \operatorname{Spec} A_{\text{inf}}$, and a map

$$Y \to |X|$$

sending $\xi \mapsto V(\Pi(\xi))$ for a certain Weierstrass product $\Pi(\xi) \in B^{\varphi = p^d}$, inducing an isomorphism

$$|Y|/\varphi^{\mathbb{Z}} \xrightarrow{\sim} |X|.$$

We conjectured that Y “has the structure of a rigid analytic space”. We also remarked that the residue fields of X are “strictly p-perfect”, which now go under the name of “perfectoid fields”, whose tilts were finite degree extensions of F.

4. Vector bundles

We proved the classification result for vector bundles over $X = \operatorname{Proj}(P)$. The notion of “modifications of vector bundles” arose naturally in the proof.

For example, when trying to classify rank 2 vector bundles, degree -1 modifications of $\mathcal{O}(1/2)$

$$0 \to \mathcal{E} \to \mathcal{O}(1/2) \to i_{\infty *} C \to 0$$

show up. We had to prove that $\mathcal{E} \simeq \mathcal{O}^2$. (This is implied by the classification result.) In fact, analogous problems had already been considered by Colmez-Fontaine in their proof of “weakly admissible implies admissible”, and it involved the so-called “fundamental lemma of p-adic Hodge theory”.

Let $(\lambda_0, \lambda_1) \in C^2 \setminus \{(0,0)\}$. Look at the $\mathbb{Q}_p$-linear map

$$(B_{\text{cris}}^+)^{\varphi^2 = p} \to C$$

given by

$$x \mapsto \lambda_0 \theta(x) + \lambda_1 \theta(\varphi(x)).$$

This θ is a quasi-logarithm, for the Dieudonné module $\langle \theta, \theta \circ \varphi \rangle$ (notice φ acts as $\begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix}$). The “fundamental lemma” says that this is surjective with kernel a 2-dimensional $\mathbb{Q}_p$-vector space.

I remarked that this is in fact a consequence of a result of Gross-Hopkins, building on work of Laffaille. They study a de Rham period morphism

$$\pi_{\text{dR}} \colon (\text{Lubin-Tate space}) \to \mathbb{P}^1$$

and show it is surjective. This surjectivity can be rephrased as saying that any line in C^2 is the Hodge filtration of of an elliptic curve over $\mathcal{O}_C$ with supersingular reduction. This surjectivity result for π_{dR} plus comparison theorems imply our result about modifications of vector bundles. I thought this was a very cool application of geometric thinking.

Idea: to prove the classification theorem we need to prove that if

$$0 \to \mathcal{O}_X^2 \to \mathcal{E} \to i_{\infty *}C \to 0$$

is a degree 1 modification of $\mathcal{O}^2$, then

$$\mathcal{E} \simeq \begin{cases} \mathcal{O} \oplus \mathcal{O}(1) \\ \mathcal{O}(1/2) \end{cases}$$

The Drinfeld upper-half space Ω shows up in the proof. The case $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{O}(1)$ shows up in relation to $\partial\Omega = \mathbb{P}^1(\mathbb{Q})$, and the case $\mathcal{E} \simeq \mathcal{O}(1/2)$ shows up in relation to Ω.

It turns out that Hodge-Tate periods of Lubin-Tate spaces, more precisely the surjectivity of

$$\pi_{\mathrm{HT}}\colon \text{infinite-level Lubin-Tate space} \to \Omega,$$

enter into the proof of the classification theorem. (Compare with the "de Rham case" where the modification appears on the other side.) Combining this with comparison theorems imply the desired results about modifications of vector bundles.

This was the first time I really saw the link between Rapoport-Zink spaces and moduli of modifications of vector bundles. [Cf. my article "Rigid analytic p-divisible groups".] In particular, Ω is interpreted as a moduli space of modifications. This is very important for the Hecke property in my Conjecture.

5. Modifications of vector bundles

The second occurrence of modifications of vector bundles was in the proof of "weakly admissible implies admissible".

We have seen that given an isocrystal (D, φ) we can attach a vector bundle $\mathcal{E}(D, \varphi)$ on the curve X.

Given a filtered φ-module $(D, \varphi, \mathrm{Fil})$ we can attach a modification $\mathcal{E}(D, \varphi, \mathrm{Fil})$ given by the Hodge filtration.[1]

Let $K/\mathbb{Q}_p$ be a discretely valued extension with perfect residue field, $C = \widehat{\overline{K}}$ and $\Gamma = \mathrm{Gal}(\overline{K}/K)$, W a finite-dimensional K-vector space.

I really tried to understand why it was a filtration that gave a modification. One would expect that lattices give modifications. I figured out that there was an equivalence

$$\{\text{Filtrations of } W\} \xrightarrow{\sim} \{\Gamma\text{-invariant } B_{\mathrm{dR}}^+\text{-lattices in } W \otimes_K B_{\mathrm{dR}}\}.$$

The functor takes

$$\mathrm{Fil} \mapsto \mathrm{Fil}^0(W \otimes_K B_{\mathrm{dR}}) = \sum_{i \in \mathbb{Z}} \mathrm{Fil}^i W \otimes t^{-i} B_{\mathrm{dR}}^+.$$

[1] In the slides of my talk at Laumon's conference, I already say "we apply a Hecke operator" to get this modification.

This clarified why a Hodge filtration on (D,φ) was equivalent to a Γ-equivariant modification of $\mathcal{E}(D,\varphi)$.

Let $G = \mathrm{GL}_n$. Consider the action of $\mathbb{G}_m$ on Gr_G by “loop rotations”, so $\lambda \in \mathbb{C}^\times$ acts on $\mathrm{Gr}_G(\mathbb{C}) = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ by $t \mapsto \lambda t$.

For μ a cocharacter of G, let Gr^μ be the open Schubert cell. There is an affine fibration $\mathrm{Gr}^\mu \to G/P_\mu$ which is $\mathbb{C}^\times$-equivariant, which satisfies $(\mathrm{Gr}^\mu)^{\mathbb{C}^\times} \xrightarrow{\sim} G/P_\mu$.

This helps to understand the preceding formula. We replace $U(1)$ by $\Gamma = \mathrm{Gal}(\overline{K}/K)$. The LHS acts by $t \mapsto \lambda t$ and the RHS acts by $\sigma(t) = \chi_p(\sigma)t$.

So the fact about Gr_G over $\mathbb{C}$ is analogous to the bijection between Γ-equivariant modifications and filtrations.

6. The special program in Toronto

At this program Peter Scholze invented diamonds. I discussed with him a question of Colmez-Fontaine. Let $F = C^\flat$. We have an isomorphsim

$$\begin{aligned} \mathfrak{m}_F &\xrightarrow{\sim} B^{\varphi=p} \\ \epsilon &\mapsto \sum_{m\in\mathbb{Z}} [\epsilon^{p^{-n}}]p^n. \end{aligned}$$

This realizes the global sections $B^{\varphi=p}$ geometrically as the “open ball”.

The question of Colmez-Fontaine was whether there was a similar geometric interpretation of $B^{\varphi=p^2}$. This is the global sections of a bundle of slope 2, which lies outside the interval $[0,1]$ of slopes of p-divisible groups.

I was at first pessimistic about this. But Peter pointed out that I proved the multiplication map

$$B^{\varphi=p} \times B^{\varphi=p} \to B^{\varphi=p^2}$$

is surjective. Picking $t_1, t_2 \in B^{\varphi=p}$ which are $\mathbb{Q}_p$-linearly independent, we have a short exact sequence

$$0 \to \mathbb{Q}_p \to B^{\varphi=p} \oplus B^{\varphi=p} \to B^{\varphi=p^2} \to 0$$

which suggested that $B^{\varphi=p^2}$ could be realized by an “algebraic space”.

7. The Hot Topics workshop at MSRI

In one of the talks, somebody asked Scholze if any perfectoid field of characteristic p is the tilt of a characteristic 0 perfectoid field. Fontaine and I knew the answer was affirmative, and moreover that up to powers of Frobenius the closed points of the Curve were the same as untilts. That suggested a new picture of the Curve as a “moduli space of untilts”.

In the modern formulation, this statement becomes “$\mathrm{Div}^1 = \mathrm{Spa}(\mathbb{Q}_p)^\diamond/\varphi^{\mathbb{Z}}$”.

8. Luminy

I organized a conference in Luminy, where Scholze asked if I could classify G-bundles on the Curve, for any reductive group $G/\mathbb{Q}_p$. He needed this for his work with Caraiani.

I answered that it should be given by $B(G)$. This was an important impetus to consider general groups. If I had restricted my attention to GL_n, I would probably never have discovered the conjecture. (In particular, one doesn't see the L-packet phenomenon and the action of $\mathcal{S}_\varphi$ in the case of GL_n.)

REMARK 8.1. Even for GL_n, one considers $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \hookrightarrow \mathrm{GL}_{n_1+n_2}$ to construct Eisenstein series; so it is necessary to consider groups of a more general form even to study GL_n.

9. The special program in Berkeley

Scholze introduced many new objects: diamonds, local Shimura varieties (generalizations of Rapoport-Zink spaces) parametrizing moduli spaces of modifications of G-bundles, ...

I decided to upgrade everything to Bun_G for general G (as opposed to GL_n).

The **key observation** was: the Drinfeld tower shows up in the Hecke property for GL_n.

The biggest mystery for me was: how do you get local Langlands from the conjectural eigensheaf $\mathcal{F}_\varphi$? In Drinfeld's context, you take the trace of Frobenius to get a function. But here we don't want to get a function. Scholze suggested taking the stalk of $\mathcal{F}_\varphi$ at the trivial G-bundle. I was not convinced at first, but what really convinced me was the perspective on Local Langlands by Kottwitz/Kaletha, which explained the link between local Langlands and isocrystals. In particular, an important fact was that if G' is an inner form of $G/\mathbb{Q}_p$, then G' is an extended pure inner form of G if and only if $G' \times X$ is a pure inner form of $G \times X$, i.e. the reductive group scheme G' over X is obtained from twisting by a G-bundle.

CHAPTER 11

The Conjectures

1. The Lubin-Tate tower ($G = \mathrm{GL}_2$)

1.1. Lubin-Tate space at infinite level. Let $K = W(\overline{\mathbb{F}}_p)[1/p]$.

The Lubin-Tate tower is a deformation space for p-divisible groups. Let $G_0/\overline{\mathbb{F}}_p$ be a connected p-divisible group of height 2 and dimension 1. This corresponds to the isocrystal $M_{G_0} = K^2$ with $\phi_{M_{G_0}} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ (in the contravariant normalization). This is associated to the vector bundle $\mathcal{E}_{G_0} = \mathcal{O}(1/2)$ on the Fargues-Fontaine curve.

We discussed that there is a perfectoid space $\mathcal{M}_{G_0,\infty}/\operatorname{Spa} K$, whose functor of points are

$$S \mapsto \left\{ \begin{array}{c} \text{modifications } \mathcal{O}^2 \to \mathcal{O}(1/2) \text{ on } \mathcal{X}_{S^\flat} \\ \text{with cokernel locally free of rank 1 over } \mathcal{O}_S \end{array} \right\}$$

At finite level the description would actually be *more* complicated – one needs to describe a $\mathbb{Z}_p$-lattice in a $\mathbb{Q}_p$-local system associated to a vector bundle by results of Kedlaya-Liu.

Warning 1.1. We have $\underline{\mathrm{Aut}}(\mathcal{O}^2) = \underline{\mathrm{GL}_2(\mathbb{Q}_p)}$, *not* GL_2 as an algebraic group. (These are sheaves on Perf, e.g. the left side is by definition $S \mapsto \mathrm{Aut}_{\mathcal{X}_S}(\mathcal{O}^2)$.)

It turns out that $\mathrm{End}(M_{G_0}) = D$ is a quaternion algebra over $\mathbb{Q}_p$, hence this acts on $\mathcal{M}_{G_0,\infty}$. So the latter has commuting actions of $\underline{\mathrm{GL}_2(\mathbb{Q}_p)}$ and $\underline{D^\times}$.

1.2. The Gross-Hopkins period map. There is a map $\pi_{\mathrm{GH}} \colon \mathcal{M}_{G_0,\infty} \to \mathbb{P}^1_K$. This is a morphism of adic spaces, but we'll just describe what it does at the level of diamonds.

The fiber of $\mathcal{O}(1/2)$ at is $i^*\mathcal{O}(1/2) = M_{G_0} \otimes_K \mathcal{O}_S$. This is because $\mathcal{O}(1/2)$ is the descent of the trivial bundle $M_{G_0} \otimes \mathcal{O}_{\mathcal{Y}_{(0,\infty)},S}$.

The space of degree 1 modifications of $\mathcal{O}(1/2)$ along $i \colon S \hookrightarrow \mathcal{X}_{S^\flat}$, i.e. sequences of the form

$$0 \to \mathcal{F} \to \mathcal{O}(1/2) \to i_* W \to 0$$

correspond to surjections

$$M_{G_0} \otimes_K \mathcal{O}_S = i^*\mathcal{O}(1/2) \twoheadrightarrow \underbrace{W}_{\text{rank } 1}.$$

The space of such modifications is therefore $\mathbb{P}(M_{G_0})_K$. This induces the *Gross-Hopkins period map*

$$\pi_{\mathrm{GH}} \colon \mathcal{M}_{G_0,\infty} \to \mathbb{P}^1_K.$$

THEOREM 1.2. *The map π_{GH} is surjective, and in fact it's a pro-étale $\mathrm{GL}_2(\mathbb{Q}_p)$-torsor.*

REMARK 1.3. This shows that $\mathbb{P}^1_K$ is not "simply connected" in the pro-étale topology!

Let's take a closer look at the modifications. Given a degree 1 modification

$$0 \to \mathcal{F} \to \mathcal{O}(1/2) \to i_* W \to 0,$$

we see that $\mathcal{F}$ is pointwise of rank 2 and degree 0. The classification theorem implies $\mathcal{F} \simeq \mathcal{O}^2$ or $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ or $\mathcal{O}(2) \oplus \mathcal{O}(-2)$, etc. But bundles can only map trivially to other bundles of lower degree, since negative slope bundles have no global sections. So in fact $\mathcal{F}$ must pointwise be $\mathcal{O}^2$. Globally, what we can say is that it is semistable of slope 0. A theorem of Kedlaya-Liu, $\mathcal{F}$ says that semistable slope 0 bundles on $\mathcal{X}_{S^\flat}$ are equivalent to rank 2 $\mathbb{Q}_p$-local systems $\mathbb{L}$ on S. (Explicitly, $\mathbb{L}$ is the space of trivializations of $\mathcal{F}$.)

1.3. The Hodge-Tate period map. Consider the space of all modifications

$$0 \to \mathcal{O}^2 \to \mathcal{E} \to i_* W \to 0.$$

We can play the same game: such a modification is determined by the kernel of

$$\mathcal{O}_S^2 = i^* \mathcal{O}^2 \to i^* \mathcal{E}$$

i.e. by a rank 1 sub-bundle of $\mathcal{O}_S^2$. This defines the *Hodge-Tate period map*

$$\pi_{\mathrm{HT}} \colon \mathcal{M}_{G_0,\infty} \to \mathbb{P}^1_K.$$

However, this is no longer surjective.

THEOREM 1.4. *The Hodge-Tate period map π_{HT} factors through $\Omega_K = \mathbb{P}^1_K \setminus \underline{\mathbb{P}^1(\mathbb{Q}_p)}$, and the map $\mathcal{M}_{G_0,\infty} \to \Omega_K$ is a pro-étale $\underline{D}^\times$-torsor.*

Given

$$0 \to \mathcal{O}^2 \to \mathcal{E} \to i_* W \to 0,$$

we deduce that $\mathcal{E}$ has rank 2 and degree 1. Pointwise $\mathcal{E}$ is $\mathcal{O}(1/2)$ or $\mathcal{O} \oplus \mathcal{O}(1)$ (as before, one checks that the other bundles in the classification can't fit into such a short exact sequence). Now we have to consider two cases:

- If $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{O}(1)$, then the map $\mathcal{O}^2 \to \mathcal{E}$ is determined by a matrix

  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}) = \mathbb{Q}_p; c, d \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) = B^{\phi=p}.$$

  Hence $ad - bc \in B^{\phi=p}$ has a zero at S. Then π_{HT} sends this point to $[-b : a] \in \mathbb{P}^1$, because π_{HT} sends a modification to the kernel of the map at S, and one easily computes that

  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ ad - bc \end{pmatrix}$$

  vanishes at S.

REMARK 1.5. For general n, we have

$$\mathbb{P}^{n-1} = \{0 \to \mathcal{O}^n \to \mathcal{E} \to i_*W \to 0\} \to \mathrm{Bun}_n^{\deg 1}.$$

The allowable $\mathcal{E}$ are

$$\mathcal{O}(1/n) \rightsquigarrow \mathcal{O} \oplus \mathcal{O}(\frac{1}{n-1}) \rightsquigarrow \mathcal{O}^2 + \mathcal{O}(\frac{1}{n-2}) \rightsquigarrow \ldots \rightsquigarrow \mathcal{O}^{n-1} \oplus \mathcal{O}(1).$$

What is the induced stratification on $\mathbb{P}^{n-1}$? The pre-image of $\mathcal{O}(1/n)$ is

$$\pi_{\mathrm{HT}}^{-1}(\mathcal{O}(1/2n)) = \Omega := \mathbb{P}^{n-1} \setminus \text{ union of } \mathbb{Q}_p\text{-rational hyperplanes.}$$

The pre-image of $\mathcal{O}^{n-1} \oplus \mathcal{O}(1)$ is

$$\pi_{\mathrm{HT}}^{-1}(\mathcal{O}^{n-1} \oplus \mathcal{O}(1)) = \mathbb{P}^{n-1}(\mathbb{Q}_p).$$

The rest of the stratification fills this in, e.g. the union of $\mathbb{Q}_p$-rational hyperplanes minus $\mathbb{Q}_p$-rational planes of codimension 2, etc.

This is a special case of the Newton stratification of Caraiani-Scholze.

Picture: $\mathcal{M}_{G_0,\infty}$ has an action of $\underline{\mathrm{GL}_2(\mathbb{Q}_p)} \times \underline{D}^\times$. The quotient by $\underline{\mathrm{GL}_2(\mathbb{Q}_p)}$ is the Gross-Hopkins period map, and makes $\mathcal{M}_{G_0,\infty}$ a pro-étale $\underline{\mathrm{GL}_2(\mathbb{Q}_p)}$-torsor over $\mathbb{P}(M_{G_0})$. On the other hand, the quotient by $\underline{D}^\times$ is the Hodge-Tate period map, and makes $\mathcal{M}_{G_0,\infty}$ a pro-étale $\underline{D}^\times$-torsor over Ω.

$$\begin{tikzcd} & \mathcal{M}_{G_0,\infty} \arrow[dl, "\pi_{\mathrm{GH}}"', "\underline{\mathrm{GL}_2(\mathbb{Q}_p)}"] \arrow[dr, "\pi_{\mathrm{HT}}", "\underline{D}^\times"'] & \\ \underline{D}^\times \curvearrowright \mathbb{P}(M_{G_0}) & & \Omega \curvearrowleft \underline{\mathrm{GL}_2(\mathbb{Q}_p)} \end{tikzcd} \tag{1.1}$$

2. Connection to Fargues' Conjecture

We have a diagram

$$\begin{tikzcd} & \mathrm{Hecke} \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow} \times \mathrm{supp}"] & \\ \mathrm{Bun}_{\mathrm{GL}_2} & & \mathrm{Bun}_{\mathrm{GL}_2} \times (\mathrm{Spa}\, K)^\diamond/\varphi^{\mathbb{Z}} \end{tikzcd}$$

Let $\phi\colon W_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be a discrete Weil parameter and let $\mathcal{F}_\phi$ be the Hecke eigensheaf in $D_{\text{ét}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ predicted by Fargues' conjecture. Applying the eigensheaf property to $\mathrm{Std} \in \mathrm{Rep}(\mathrm{GL}_2)$, we have $\mathrm{Sat}(\mathrm{Std}) = \mathbb{Q}_\ell|_{\mathrm{Hecke}^1}[1]$.

Since the Satake sheaf is supported on Hecke^1, we might as well replace Hecke by Hecke^1.

$$\begin{tikzcd} & \mathrm{Hecke}^1 \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow} \times \mathrm{supp}"] & \\ \mathrm{Bun}_{\mathrm{GL}_2}^{\deg 1} & & \mathrm{Bun}_{\mathrm{GL}_2}^{\deg 0} \times (\mathrm{Spa}\, K)^\diamond/\varphi^{\mathbb{Z}} \end{tikzcd}$$

The eigensheaf property says that

$$(h^{\rightarrow} \times \mathrm{supp})_!(h^{\leftarrow *}\mathcal{F}_\phi) \simeq \mathcal{F}_\phi \boxtimes \phi. \tag{2.1}$$

Consider the fibered product

$$\begin{tikzcd}
? \arrow[r] \arrow[d] & {[*/\underline{\mathrm{GL}_2(\mathbb{Q}_p)}] \times (\mathrm{Spa}\, K)^\diamond/\varphi^{\mathbb{Z}}} \arrow[d, "(x_1 \times \mathrm{Id})"] \\
\mathrm{Hecke}^1 \arrow[r, "h^{\rightarrow}\times \mathrm{supp}"] & \mathrm{Bun}_{\mathrm{GL}_2}^{\deg 0} \times (\mathrm{Spa}\, K)^\diamond/\varphi^{\mathbb{Z}}
\end{tikzcd}$$

The fibered product is essentially just the space of degree 1 modifications of the trivial bundle (up to quotienting by the action of $\underline{\mathrm{GL}_2(\mathbb{Q}_p)}$ and $\varphi^{\mathbb{Z}}$). We just saw that this is $\mathbb{P}^1$. So the fibered product is

$$\begin{tikzcd}
(\mathbb{P}^1_K/\underline{\mathrm{GL}_2(\mathbb{Q}_p)})/\varphi^{\mathbb{Z}} \arrow[r, "(h^{\rightarrow})'"] \arrow[d, "i"] & {[*/\underline{\mathrm{GL}_2(\mathbb{Q}_p)}] \times (\mathrm{Spa}\, K)^\diamond/\varphi^{\mathbb{Z}}} \arrow[d, "x_1 \times \mathrm{Id}"] \\
\mathrm{Hecke}^1 \arrow[r, "h^{\rightarrow}\times \mathrm{supp}"] & \mathrm{Bun}_{\mathrm{GL}_2}^{\deg 0} \times (\mathrm{Spa}\, K)^\diamond/\varphi^{\mathbb{Z}}
\end{tikzcd} \tag{2.2}$$

Pulling back (2.1) through $(x_1, \mathrm{Id})^*$ gives

$$(x_1, \mathrm{Id})^*(h^{\rightarrow} \times \mathrm{supp})_! h^{\leftarrow *}\mathcal{F}_\phi[1] \simeq (x_1, \mathrm{Id})^*(\mathcal{F}_\phi \boxtimes \phi) \tag{2.3}$$

By definition, $x_1^*(\mathcal{F}_\phi) = \pi_\phi$. So the RHS of (2.3) is $(x_1, \mathrm{Id})^*(\mathcal{F}_\phi \boxtimes \phi) \simeq \pi_\phi \boxtimes \phi$. By proper base change we may rewrite the LHS of (2.1) as:

$$(x_1, \mathrm{Id})^*(\mathcal{F}_\phi \boxtimes \phi) \simeq (h^{\rightarrow})'_! i^* h^{\leftarrow *}\mathcal{F}_\phi[1] \simeq (h^{\rightarrow})'_!(h^{\leftarrow} \circ i)^*\mathcal{F}_\phi[1]. \tag{2.4}$$

Assume ϕ is irreducible. Then Fargues' Conjecture implies that $\mathcal{F}_\phi$ is supported on the semistable locus: $\mathcal{F}_\phi = j_! j^* \mathcal{F}_\phi$ where $j \colon \mathrm{Bun}_{\mathrm{GL}_2}^{ss, \deg 1} \hookrightarrow \mathrm{Bun}_{\mathrm{GL}_2}^{\deg 1}$. We know that $\mathrm{Bun}_{\mathrm{GL}_2}^{ss,\deg 1} = [*/\underline{J_b(\mathbb{Q}_p)}]$, so a sheaf on it can be identified with a representation of $J_b(\mathbb{Q}_p)$. In other words, $\mathcal{F}_\phi = j_!(\pi_\phi)_b$.

Putting this into the RHS of (2.4), we get

$$(h^{\rightarrow})'_! i^* h^{\leftarrow *}\mathcal{F}_\phi[1] \simeq (h^{\rightarrow})'_!(h^{\leftarrow} \circ i)^*\mathcal{F}_\phi[1] \simeq (h^{\rightarrow})'_!(h^{\leftarrow} \circ i)^* j_!((\pi_\phi)_b)[1]. \tag{2.5}$$

Now we look at the LHS of (2.6)

$$\begin{tikzcd}
 & {[\mathbb{P}^1_K/\underline{\mathrm{GL}_2(\mathbb{Q}_p)}]/\varphi^{\mathbb{Z}}} \arrow[d, "i"] \\
 & \mathrm{Hecke}^1 \arrow[d, "h^{\leftarrow}"] \\
{[*/\underline{J_b(\mathbb{Q}_p)}]} \arrow[r, hook, "j"] & \mathrm{Bun}_{\mathrm{GL}_2}^{\deg 1}
\end{tikzcd} \tag{2.6}$$

The right vertical composition map sends "degree 1 modifications of the trival bundle to the modification". So the fibered product is

$$\begin{tikzcd} {[\Omega^1_K/\underline{\mathrm{GL}_2(\mathbb{Q}_p)}]/\varphi^{\mathbb{Z}}} \arrow[r, "j'"] \arrow[d, "\alpha"] & {[\mathbb{P}^1_K/\underline{\mathrm{GL}_2(\mathbb{Q}_p)}]/\varphi^{\mathbb{Z}}} \arrow[d, "h^{\leftarrow}\circ i"] \\ {[*/\underline{J_b(\mathbb{Q}_p)}]} \arrow[r, hook, "j"] & \mathrm{Bun}^{\deg 1}_{\mathrm{GL}_2} \end{tikzcd} \tag{2.7}$$

By proper base change applied to (2.7), the RHS of (2.5) can be rewritten as

$$(h^{\rightarrow})'_!(h^{\leftarrow}\circ i)^* j_!((\pi_\phi)_b)[1] \simeq (h^{\rightarrow})'_!(j'_!)\alpha^*((\pi_\phi)_b)[1]. \tag{2.8}$$

Let's combine this with more of (2.2)

$$\begin{tikzcd} {[\Omega^1_K/\underline{\mathrm{GL}_2(\mathbb{Q}_p)}]/\varphi^{\mathbb{Z}}} \arrow[r, "j'"] \arrow[d, "\alpha"] & {[\mathbb{P}^1_K/\underline{\mathrm{GL}_2(\mathbb{Q}_p)}]/\varphi^{\mathbb{Z}}} \arrow[r, "(h^{\rightarrow})'"] \arrow[d, "h^{\leftarrow}\circ i"] & {[*/\underline{\mathrm{GL}_2(\mathbb{Q}_p)}] \times (\mathrm{Spa}\, K)^{\diamond}/\varphi^{\mathbb{Z}}} \\ {[*/\underline{J_b(\mathbb{Q}_p)}]} \arrow[r, hook, "j"] & \mathrm{Bun}^{\deg 1}_{\mathrm{GL}_2} & \end{tikzcd} \tag{2.9}$$

The composition along the top row,

$$(h^{\rightarrow})'\circ j' \colon [\Omega^1_K/\underline{\mathrm{GL}_2(\mathbb{Q}_p)}]/\varphi^{\mathbb{Z}} \to [*/\underline{\mathrm{GL}_2(\mathbb{Q}_p)}] \times (\mathrm{Spa}\, K)^{\diamond}/\varphi^{\mathbb{Z}}$$

is basically a structure morphism. Let's try to under the left vertical map in (2.9). We saw that the Hodge-Tate period map induced an isomorphism

$$\mathcal{M}_{G_0,\infty}/(\underline{\mathrm{GL}_2(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)}) \xrightarrow{\sim} [\Omega^1_K/\underline{\mathrm{GL}_2(\mathbb{Q}_p)}].$$

Refer to (1.1): applying the Gross-Hopkins period map effects quotienting out in the other order:

$$\begin{tikzcd} \mathcal{M}_{G_0,\infty}/(\underline{\mathrm{GL}_2(\mathbb{Q}_p)} \times \underline{J_b(\mathbb{Q}_p)}) \arrow[r, "\pi_{\mathrm{HT}}", "\sim"'] \arrow[d, "\pi_{\mathrm{GH}}", "\sim"'] & {[\Omega^1_K/\underline{\mathrm{GL}_2(\mathbb{Q}_p)}]} \arrow[d, "\alpha"] \\ {[\mathbb{P}^1/\underline{J_b(\mathbb{Q}_p)}]} \arrow[r] & {[*/\underline{J_b(\mathbb{Q}_p)}]} \end{tikzcd}$$

and we can now interpret the bottom horizontal map also as a structure map, so pushforward along amounts to taking cohomology. Replacing $(h^{\rightarrow})'_! \circ j'_!$ with this structure morphism, we can rewrite the RHS of (2.8) as

$$(h^{\rightarrow})'_!(j'_!)\alpha^*((\pi_\phi)_b)[1] \simeq \left(R\Gamma_c(\mathcal{M}_{G_0,\infty}, \overline{\mathbb{Q}}_\ell)[1] \otimes (\pi_\phi)_b\right)^{J_b(\mathbb{Q}_p)}$$

and by (2.1), it is equated with $\pi_\phi \boxtimes \phi$.

Another way to write this is:

$$H^1_c(\mathcal{M}_{G_0,\infty}; \overline{\mathbb{Q}}_\ell)^{\mathrm{cusp}} \simeq \bigoplus_{\pi \in \mathcal{A}_2^{\mathrm{cusp}}} \pi \otimes \pi_b^{\vee} \boxtimes \phi_\pi.$$

This is a theorem of Deligne-Carayol. The higher rank version, for GL_n, is due to Harris-Taylor.

3. How to do automorphic to Galois?

3.1. Number fields vs function fields. We briefly survey the analogy between number fields and function fields.

3.1.1. $G = \mathrm{GL}_1$. For $F = \mathbb{Q}$, one generates abelian extensions by adjoining division points of $\mathbb{G}_m$. There is a uniformization

$$\mathbb{C}/2\pi i\mathbb{Z} \xrightarrow{\exp} \mathbb{G}_m(\mathbb{C}).$$

For $F = \mathbb{F}_q(T)$ and $A = \mathbb{F}_q[T]$, one generates abelian extensions by adjoining division points of the Carlitz module M. With $C = \widehat{\overline{F}}_\infty$, there is a uniformization

$$C \xrightarrow{\exp} C/\xi A = M(C).$$

For $F/\mathbb{Q}$ an imaginary quadratic extension, one generates abelian extension by adjoining division points of CM elliptic curves. For general function fields F, one can generate abelian extension by adjoining division points of Drinfeld A-modules of rank 1.

3.1.2. $G = \mathrm{GL}_2$. Over $F = \mathbb{Q}$, one finds the Langlands correspondence in the cohomology of moduli spaces of elliptic curves. Over function fields, one finds the Langlands correspondence in the cohomology of moduli of Drinfeld A-modules of rank 2.

The function-field picture generalizes well to Drinfeld A-modules of rank d. These generalize to Drinfeld shtukas for GL_d with a pole at ∞ and varying 0. These generalize further to arbitrary reductive G. For number fields one has moduli of abelian varieties, but these encompass allow access to a much more restricted selection of groups g.

3.2. Drinfeld's shtukas. Fix $X/\mathbb{F}_q$ a smooth projective curve.

Definition 3.1. Let S an $\mathbb{F}_q$-scheme. Let I be a finite set. An *X-shtuka over S of rank n with legs $x_i \in X(S)$* is the datum of:

- a vector bundle $\mathcal{E}$ on $X \times_{\mathbb{F}_q} S$ of rank n,
- an isomorphism

$$\phi_{\mathcal{E}} \colon (\mathrm{Id} \times \mathrm{Frob}_S)^* \mathcal{E}|_{X \times S \setminus \bigcup \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}_{X \times S \setminus \bigcup \Gamma_{x_i}}.$$

Let $\mathrm{Sht}_{X,I}$ be the moduli stack of these, which has a map $\mathrm{Sht}_{X,I} \to X^I$ by projecting to the x_i.

The spaces $\mathrm{Sht}_{X,I}$ are quite large (infinite type and infinite dimensional). To cut them down, we impose bounds on the poles of $\phi_{\mathcal{E}}$. For $G = \mathrm{GL}_n$, each $W \in \mathrm{Rep}(\widehat{G}^I)$ gives a bounded version $\mathrm{Sht}_{X,I,W}$ which is a Deligne-Mumford stack.

Example 3.2. Drinfeld studied the case $G = \mathrm{GL}_2$, $I = \{1, 2\}$, $W = \mathrm{Std} \boxtimes \mathrm{Std}^\vee$ to prove global Langlands for GL_2 over function fields.

Laurent Lafforgue studied the case $G = \mathrm{GL}_n$, $I = \{1, 2\}$, $W = \mathrm{Std} \boxtimes \mathrm{Std}^\vee$ to prove global Langlands for GL_n over function fields.

Vincent Lafforgue studied general G, I, W to get the automorphic-to-Galois direction in general.

We'd like to transport this to the situation of local fields.

3.3. p-adic fields. We try to replicate this for X replaced by "$(\mathrm{Spa}\,\mathbb{Z}_p)^\diamond$". Recall that for $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}$, we defined

$$\mathcal{Y}_{[0,\infty),S} = \mathrm{Spa}\, W(R^+) \setminus \{\varpi = 0\}.$$

Recall the diamond formula

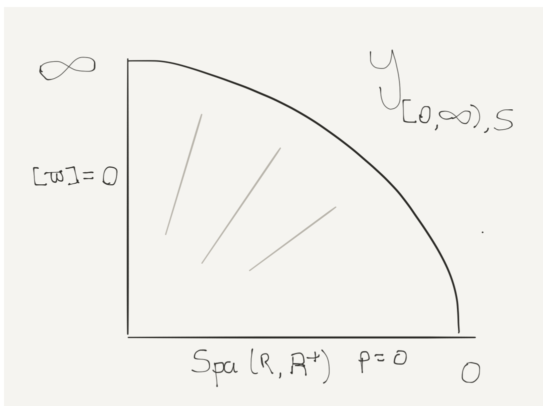
$$\mathcal{Y}^\diamond_{[0,\infty),S} = X \times S^\diamond.$$

DEFINITION 3.3 (Mixed characteristic shtukas). Let $S \in \mathrm{Perf}$ and $S_i^\#$ be an untilt of S over $\mathbb{Q}_p$ for all $i \in I$. This gives a collection of divisors $S_i^\# \hookrightarrow \mathcal{Y}_{(0,\infty),S}$. A *shtuka* is the datum of:

- a rank n vector bundle $\mathcal{E}$ on $\mathcal{Y}_{[0,\infty),S}$,
- a meromorphic isomorphism

$$\phi_{\mathcal{E}} \colon \mathrm{Frob}_S^* \mathcal{E}|_{\mathcal{Y}_{[0,\infty),S} \setminus \bigcup_i S_i^\#} \xrightarrow{\sim} \mathcal{E}|_{\mathcal{Y}_{[0,\infty),S} \setminus \bigcup_i S_i^\#}.$$

Picture of $\mathcal{Y}_{[0,\infty),S} \subset \mathrm{Spa}\, W(R^+)$:



EXAMPLE 3.4. If there are no legs, i.e. $I = \emptyset$, then $(\mathcal{E}, \phi_{\mathcal{E}})$ corresponds to a $\mathbb{Z}_p$-local system $\mathbb{L}$ on S. This is a theorem of Kedlaya; the inverse correspondence takes $\mathbb{L} \mapsto \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{Y}_{[0,\infty),S}}$ (Kedlaya-Liu).

A pair $(\mathcal{E}, \phi_{\mathcal{E}})$ on $\mathcal{Y}_{(0,\infty),S} \subset \mathcal{Y}_{[0,\infty),S}$ with no legs is the same as a vector bundle on $\mathcal{X}_S$.

Given a shtuka, we get a vector bundle on $\mathcal{X}_S$ by "looking near ∞". By looking near 0, we get a $\underline{\mathbb{Z}_p}$-local system, hence a semistable bundle of slope 0. So $\mathrm{Sht}_{G,b}$ can be described as shtukas with an isomorphism $\mathcal{E}_\infty \simeq \mathcal{E}_0$. By adding a trivialization of $\mathcal{E}_0$, we get the infinite-level moduli space of shtukas $\mathrm{Sht}_{G,b,I,\infty} \to \mathrm{Sht}_{G,b,I}$.

For $W \in \mathrm{Rep}(\widehat{G}^I)$, we get a bounded version $\mathrm{Sht}_{G,b,W,\infty}$.

THEOREM 3.5 (Scholze). $\mathrm{Sht}_{G,b,W,\infty}$ *is a locally spatial diamond.*

REMARK 3.6. This space $\mathrm{Sht}_{G,b,I,W,\infty}$ has a $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W^I_{\mathbb{Q}_p}$- action.

The proof works by considering a period map to an affine Grassmannian, which is a pro-étale torsor onto its image, thus reducing to the theorem to a statement for the affine Grassmannian.

EXAMPLE 3.7. The Lubin-Tate tower at infinite level is a special case:

$$\mathrm{Sht}_{\mathrm{GL}_2,\begin{pmatrix}0 & 1\\ p & 0\end{pmatrix},\{1\},\mathrm{Std}_\infty} \simeq \mathcal{M}^{\diamond}_{G_0,\infty}.$$

3.4. Kottwitz' Conjecture. Given $\phi\colon W_{\mathbb{Q}_p} \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$. Conjecturally we have associated representations $\pi_\phi, (\pi_\phi)_b$. Then we decompose the action

$$G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p) \times W^I_{\mathbb{Q}_p} \curvearrowright R\Gamma_c(\mathrm{Sht}_{G,b,W,\infty}; \mathrm{Sat}(W))_{\mathrm{cusp}}$$

into irreducibles. Kottwitz' Conjecture predicts that this decomposition has the form

$$R\Gamma_c(\mathrm{Sht}_{G,b,W,\infty}; \mathrm{Sat}(W))_{\mathrm{cusp}} \simeq \bigoplus_{\phi \text{ irred}} \pi_\phi \boxtimes (\pi_\phi)^\vee_b \boxtimes (W \otimes \phi^I)$$

where $W \circ \phi^I$ is the inflated Galois representation

$$W^I_{\mathbb{Q}_p} \xrightarrow{\phi^I} {}^L G^I \xrightarrow{W} \mathrm{GL}_N(\overline{\mathbb{Q}}_\ell).$$