Siegel modular varieties and Borel-Serre compactification of modular curves

Michele Fornea

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1 Scholze's Theorem on Torsion classes

To motivate the study of Siegel modular varieties and Borel-Serre compactifications let me recall Scholze's theorem on torsion classes.
We start from an Hecke eigenclass $h \in H^i(X_g/\Gamma, \mathbb{F}_p)$ where X_g is the locally symmetric domain for GL_g, and we want to attach to it a continuous semisimple Galois representation $\rho_h : G_{\mathbb{Q}} \to \mathrm{GL}_g(\overline{\mathbb{F}}_p)$ such that the characteristic polynomials of the Frobenius classes at unramified primes are determined by the Hecke eigenvalues of h.
Problem: X_g/Γ is not algebraic in general, in fact it could be a real manifold of odd dimension; while the most powerful way we know to construct Galois representations is by considering étale cohomology of algebraic varieties.
Idea(Clozel): Find the cohomology of X_g/Γ in the cohomology of the boundary of the Borel-Serre compactification $\overline{\mathbf{A}}_g^{\mathrm{BS}}$ of Siegel modular varieties.

1.0.1 Properties of the Borel-Serre compactification

- $\overline{\mathbf{A}}_g^{\mathrm{BS}}$ is a real manifold with corners.
- The inclusion $\mathbf{A}_g \hookrightarrow \overline{\mathbf{A}}_g^{\mathrm{BS}}$ is an homotopy equivalence (same cohomology).
- The boundary of $\overline{\mathbf{A}}_g^{\mathrm{BS}}$ is parametrized by parabolic subgroups of Sp_{2g} and consists of torus bundles over arithmetic domains for the Levi subgroups of each parabolic.

It follows from the excision exact sequence for the couple $(\overline{\mathbf{A}}_g^{\mathrm{BS}}, \partial)$ that one can associate to the Hecke eigenclass $h \in H^i(X_g/\Gamma, \mathbb{F}_p)$ another eigenclass $h' \in H^i(\mathbf{A}_g, \mathbb{F}_p)$ whose eigenvalues are precisely related to those of h. Scholze's main contribution comes in the form of the following theorem.

Theorem 1.1 (Scholze). *There exists a Siegel eigencusp form f of genus g whose eigenvalues are those of h' modulo p.*

Now we are in good shape because we know how to associate Galois representations to Siegel modular forms.

Theorem 1.2. *If f is Siegel eigencusp form of genus g, then for every prime p there exists a continuous semisimple Galois representation*

$$\rho_g : \ G_{\mathbb{Q}} \longrightarrow GL_{2g+1}(\overline{\mathbb{Q}}_p)$$

characterised by the eigenvalues of f.

Finally, as the eigenvalues modulo p of the Siegel modular form f "come from the boundary", we have

$$\overline{\rho}_f \equiv \rho_h \oplus \rho_h^\vee \oplus \mathbb{I} \mod p$$

for some representation $\rho_h : G_{\mathbb{Q}} \to \mathrm{GL}_g(\overline{\mathbb{F}}_p)$ which turns out to be the representation we where looking for.

2 Siegel Modular Varieties

Siegel modular varieties are moduli space of abelian varieties endowed with a principal polarization. They are a generalization of modular curves as elliptic curves have a unique principal polarization. We would like to think about them as parametrizing all abelian varieties, however even if the level structure we will add to rigidify the moduli problem always exists after some finite étale base change, it is not true that every abelian variety has a principal polarization. What is true is that every abelian variety has a polarization, being projective, so one would need to relax the requirement of being principally polarized.
To define the moduli problem we need a good notion of families of abelian varieties. This is achieved in the following definition of abelian schemes.

Definition 2.1. *An abelian scheme $A \to S$ is a group scheme which is smooth, proper with geometrically connected fibers.*

We introduce the notion of dual abelian scheme to later define polarizations. Consider the functor

$$\underline{\mathrm{Pic}}(A/S) : \ \mathrm{Sch}/S \longrightarrow \mathrm{Sets}$$

which maps a scheme $T \to S$ to isomorphism classes of invertible sheaves $\mathcal{L}$ on $A \times_S T$ with a rigidification $\xi : \ O_T \xrightarrow{\sim} e_T^* \mathcal{L}$.

Fact 2.1. *$\underline{Pic}(A/S)$ is representable by a reduced group scheme $Pic(A/S)$.*

Denote by $A^\vee$ the connected component of the identity of $\mathrm{Pic}(A/S)$. It is an abelian scheme and we call it the dual abelian scheme of A/S.
Given an invertible sheaf $\mathcal{L}$ on A we can define a morphism

$$\lambda(\mathcal{L}) : \ A \longrightarrow A^\vee$$

by sending a functorial point $(a : Z \to A) \in A(Z)$ to the invertible sheaf $T_a^* \mathcal{L} \otimes \mathcal{L}^{-1} \otimes a^* \mathcal{L}^{-1} \otimes \epsilon^* \mathcal{L} \in A^\vee(Z)$.

Remark 2.1. *There is a heavy abuse of notation in the definition of $\lambda(\mathcal{L})$. Make sure you uderstand how it should be defined precisely and why it has a natural rigidification. In this way you know that there is a map $\lambda(\mathcal{L}) : A \to Pic(A/S)$; you should check that it maps zero to zero to first deduce that the image is contained in $A^\vee$ because A is geometrically connected and finally that it is a group homomorphism by rigidity of abelian schemes.*

Definition 2.2. *A principal polarization of A/S is an isomorphism $\lambda : A \to A^\vee$ such that for each geometric point $\bar{s} \in S$, $\lambda_{\bar{s}} = \lambda(\mathcal{L}_{\bar{s}})$ for some ample invertible sheaf $\mathcal{L}_{\bar{s}}$ on $A_{\bar{s}}$.*

2.0.1 Moduli problem

Let g, N be positive integers and consider the functor

$$\mathcal{A}_{g,N} : \ \mathrm{Sch}_{\mathbb{Z}[1/N]} \longrightarrow \mathrm{Sets}$$

which maps a scheme S where N is invertible to isomorphism classes of triples $(A, \phi_N, \lambda)_{/S}$ where

- A/S is an abelian scheme of relative dimension g.
- $\phi_N : (\mathbb{Z}/N\mathbb{Z})^{2g}_{/S} \xrightarrow{\sim} A[N]_{/S}$ is an isomorphism of group schemes over S.
- $\lambda : A \to A^\vee$ is a principal polarization.

Remark 2.2. *The group scheme $A[N]_{/S}$ of N-torsion points of an abelian scheme A/S is always finite locally free because $[N]_{A/S} : A \to A$ is an isogeny (i.e. a quasi-finite surjective group homomorphism) as one can check fiber by fiber; hence finite and flat. Under the additional assumption that N is invertible over S the group scheme $A[N]_{/S}$ is also étale, as fiber by fiber the morphism $[N]_{A/S}$ is unramified. In particular we always have an isomorphism $(\mathbb{Z}/N\mathbb{Z})^{2g}_{/S'} \cong A[N]_{/S'}$ for some $S' \to S$ finite étale.*
It follows that if we want our abelian schemes to have a level N structure ϕ_N it is necessary to work over $Spec(\mathbb{Z}[1/N])$.

Theorem 2.1 (Mumford)**.** *If $N \geq 3$, then $\mathcal{A}_{g,N}$ is representable by a smooth quasi-projective scheme $A_{g,N} \to Spec(\mathbb{Z}[1/N])$.*

A non-trivial (at least for $g \geq 2$) straightforward corollary of Mumford's representability theorem is the following corollary.

Corollary 2.1. *Every (principally polarized) abelian variety in characteristic p can be lifted to characteristic zero.*

As $A[N]_{/S}$ is finite locally free, it has a Cartier dual $A[N]^D_{/S}$ defined on $\mathrm{Sch}_{/S}$ by

$$A[N]^D(T) = \mathrm{Hom}_{T\mathrm{GrSch}}(A[N]_{/T}, \mathbb{G}_{m/T}).$$

There is a perfect pairing $A[N] \times_S A[N]^D \to \mathbb{G}_{m/S}$ and we have an identification $A[N]^D_{/S} \cong A^\vee[N]_{/S}$. Therefore, using a principal polarization $\lambda : A \xrightarrow{\sim} A^\vee$, we have constructed a Weil pairing

$$e^\lambda_{A,N} : \ A[N] \times_S A[N] \to \mu_{N/S}$$

which is alternating and perfect.
Fix ζ_N a primitive N-th root of unity, $J \in M_{2g}(\mathbb{Z}/N\mathbb{Z})$ a non-degenerate alternating matrix. We define the following subfunctor

$$\mathcal{A}^J_{g,N} : \ \mathrm{Sch}_{\mathbb{Z}[\frac{1}{N}, \zeta_N]} \longrightarrow \mathrm{Sets}$$

by mapping S to isomorphism classes of triples $(A, \phi_N, \lambda)_{/S}$ as before for which moreover the following diagram commutes:

$$\begin{tikzcd} A[N] \times A[N]_{/S} \arrow[r, "e^{\lambda}_{A,N}"] & \mu_{N/S} \arrow[d, equal] \\ (\mathbb{Z}/N\mathbb{Z})^{2g}_{/S} \times (\mathbb{Z}/N\mathbb{Z})^{2g}_{/S} \arrow[u, "(\phi_N,\phi_N)"] \arrow[r, "\alpha_J"] & \mu_{N/S} \end{tikzcd}$$

where $\alpha_J(x, y) = \zeta_N^{x^t Jy}$.

Remark 2.3. *One can prove that the functor $\mathcal{A}^J_{g,N}$ is representable by a smooth quasi-projective scheme $A^J_{g,N} \to$ Spec$(\mathbb{Z}[1/N, \zeta_N])$ by noting that $\mathcal{A}^J_{g,N}$ an open subfunctor of a representable functor and thus representable.*

Definition 2.3. *The Siegel modular variety $\mathbf{A}_{g,N} \to$ Spec$(\mathbb{Z}[1/N, \zeta_N])$ classifying g-dimensional abelian varieties with full level N structure is defined to be $A^{J_g}_{g,N}$ where $J_g = \begin{pmatrix} \mathbb{O}_g & -\mathbb{I}_g \\ \mathbb{I}_g & \mathbb{O}_g \end{pmatrix}$.*

Remark 2.4. *We could obtain a canonical model of Siegel modular varieties over Spec$(\mathbb{Z}[1/N])$ by considering symplectic level N structures*

$$\varphi_N : \mu_N^g \times_S (\mathbb{Z}/N\mathbb{Z})^g \xrightarrow{\sim} A[N]_{/S}$$

because then the pairing induced by Cartier duality $\mu_N^g \times_S (\mathbb{Z}/N\mathbb{Z})^g \to \mu_{N/S}$ would not require us to choose a primitive N-th root of unity.

3 Algebraic Groups

In this section we recall a little of the theory of algebraic groups that is needed to talk about the Borel-Serre compactification.
Let K be a field of characteristic zero, $\mathbb{G}/K$ a smooth algebraic group which in this setting is the same as a reduced affine group scheme of finite type over K.

Definition 3.1. *The unipotent radical $\mathbb{R}_u\mathbb{G}$ is the maximal connected normal closed subgroup of $\mathbb{G}$ such that its $\overline{K}$-points are unipotent.*
The (solvable) radical $\mathbb{R}\mathbb{G}$ is the maximal connected normal closed subgroup of $\mathbb{G}$ such that its group of $\overline{K}$-points is solvable.

Remark 3.1. *Every smooth algebraic group over K is linear, that is, it has a faithful representation in some $GL_{n/K}$. For a $\overline{K}$-point to be unipotent we mean that it correponds to a unipotent matrix under some faithful representation.*

We call $\mathbb{G}$ reductive if the unipotent radical is trivial, $\mathbb{R}_u\mathbb{G} = \mathrm{Spec}K$, and semi-simple if the radical is trivial, $\mathbb{R}\mathbb{G} = \mathrm{Spec}K$. As unipotent groups are solvable, being semi-simple implies being reductive.
Fact: For any $\mathbb{G}/K$, there exists a maximal reductive subgroup $\mathbb{L}$ of $\mathbb{G}$ called Levi subgroup such that

$$\mathbb{G} = \mathbb{L} \cdot \mathbb{R}_u\mathbb{G} = \mathbb{R}_u\mathbb{G} \cdot \mathbb{L}.$$

This fact is relevant for us because in the boundary, of the Borel-Serre compactification of the locally symmetric space for some semi-simple group $\mathbb{G}$, appear torus bundles of locally symmetric domains of the Levi $\mathbb{L}_{\mathbb{P}}$ for each parabolic subgroup $\mathbb{P}$ of $\mathbb{G}$.

Definition 3.2. *A closed subgroup $\mathbb{P}$ of a reductive group $\mathbb{G}$ is parabolic if the quotient scheme $\mathbb{G}/\mathbb{P}$ is projective.*

Example 3.1. *The upper triangular matrices $\mathbb{B}_2 \leq SL_2$ form a parabolic subgroup. Indeed SL_2 is semi-simple and the natural transformation*

$$\mathbb{B}_2 \backslash SL_2(R) \longrightarrow \mathbb{P}^1(R)$$

sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *to $[a : c]$ is an isomorphism.*

Let's now prove a proposition that gives us two important families of reductive and semi-simple groups.

Proposition 3.1. *$GL_{n/K}$ is reductive but not semi-simple.*

Proof. The center of GL_n is $\mathbb{G}_m$, the subgroup of scalar matrices. It's connected, commutative (so solvable), normal and closed, thus $\mathbb{G}_m \leq \mathbb{R}\mathrm{GL}_n$ which implies that GL_n is not semi-simple. To prove it is reductive let's note the following facts.
Fact 1: $\mathbb{B}_n \leq \mathrm{GL}_n$ the subgroup of upper triangular matrices is a Borel subgroup, that is, a maximal, closed, connected subgroup such that $\mathbb{B}_n(\overline{K})$ is solvable. It's clearly closed, it is connected because $\mathbb{B}_n \cong \mathbb{G}_m^n \times \mathbb{A}^{n(n-1)/2}$ and $\mathbb{B}_n(\bar{K})$ is solvable for we have the following exact sequence

$$1 \to \mathbb{U}_n(\bar{K}) \to \mathbb{B}_n(\bar{K}) \to \mathbb{G}_m^n(\bar{K}) \to 1.$$

We do not prove maximality.
Fact 2: All rational Borel subgroups of a given group are conjugated under $\mathrm{GL}_n(K)$.
It follows that $\mathbb{R}\mathrm{GL}_n \leq \cap_{\text{rational Borel}}\mathbb{B}$, in particular the radical is contained in the intersection of upper and lower triangular matrices, $\mathbb{R}\mathrm{GL}_n \leq \mathbb{B}_n \cap \mathbb{B}_n^T = \mathbb{D}_n$ which are the diagonal matrices. The only normal subgroup of GL_n contained in $\mathbb{D}_n$ is the subgroup of scalar matrices, therefore $\mathbb{R}\mathrm{GL}_n = \mathbb{G}_m$. Finally $\mathbb{R}_u\mathrm{GL}_n = (\mathbb{G}_m)_u = \mathrm{Spec}K$ because the only unipotent element of a field is the identity. □

Corollary 3.1. *$SL_{n/K}$ is semisimple.*

Proof. As SL_n is a closed subgroup of GL_n we have that $\mathbb{R}\mathrm{SL}_n \leq \mathbb{R}\mathrm{GL}_n$. Furthermore, $\mathbb{R}\mathrm{GL}_n \cap \mathrm{SL}_n = \mu_n$ and we can deduce that $\mathbb{R}\mathrm{SL}_n$ is a subgroup of μ_n. The group scheme $\mu_{n/K}$ is étale because the characteristic of K does not divide n and since $\mathbb{R}\mathrm{SL}_n$ is a connected subgroup of μ_n we conclude that $\mathbb{R}\mathrm{SL}_n = \mathrm{Spec}K$. □

4 (Rational) Borel-Serre compactification of modular curves

We are going to present the Borel-Serre compactification for the group $\mathrm{SL}_{2/\mathbb{Q}}$, even if the construction is much more general and apply to all semisimple groups

[5, p.301]. The main reason for this choice is that $\mathrm{SL}_{2/\mathbb{Q}}$ is small enough to have only Borel subgroups as parabolic subgroups, but still interesting enough as modular curves arise as quotients of its locally symmetric domain.
We choose the upper-half plane $\mathfrak{H}$ as model for the locally symmetric domain $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ because it is easier to visualize its arithmetic quotients.
The two main steps of the construction are the followings:
Step 1: Construct the bordification $\overline{\mathfrak{H}}^{\mathrm{BS}}$.
Step 2: Show that the $\mathrm{SL}_2(\mathbb{Q})$-action on $\mathfrak{H}$ extends to a continuous action on the bordification $\overline{\mathfrak{H}}^{\mathrm{BS}}$.
The subgroup of upper-triangular matrices $\mathbb{B}_2$ is the standard Borel of SL_2. To visualize its role in the bordification of the upper-half plane, you could think about it as sitting on top ∞ as the upper-triangular matrices are the stabilizer of ∞, when $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{R}\cup\{\infty\}$ via Moebius transformations.
Denote by $\mathrm{B}_2 = \mathbb{B}_2(\mathbb{R})$ the real points of the standard Borel. It has a rational Langlands decomposition [5, III.1.3, p.273]

$$\mathrm{N}\times\mathrm{A}\times\mathrm{M} \xrightarrow{\sim} \mathrm{B}_2$$

given by $(\mathrm{n},\mathrm{a},\mathrm{m})\mapsto \mathrm{nam}$, where

$$\mathrm{N} = \left\{\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} : x\in\mathbb{R}\right\}, \qquad \mathrm{A} = \left\{\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix} : a\in\mathbb{R}_{>0}\right\}$$

and $\mathrm{M} = \{\pm\mathbb{I}_2\}$.
Now, B_2 acts on $\mathfrak{H}$ transitively and the Iwasawa decomposition tells us that $\mathrm{SL}_2(\mathbb{R}) = \mathrm{B}_2\cdot\mathrm{SO}_2(\mathbb{R})$. Therefore we can talk about the horospherical decomposition of a locally symmetric domain [5, III.1.4, p.273], which in our case is the usual decomposition into real and imaginary part:

$$\begin{aligned}\mathfrak{H} &\cong \mathrm{B}_2/\mathrm{B}_2\cap\mathrm{SO}_2(\mathbb{R}) = \mathrm{B}_2/\mathrm{M}\\ &\cong \mathrm{N}\times\mathrm{A}\times\{\infty\}\\ &\cong \mathbb{R}\times\mathbb{R}_{>0}\times\{\infty\}\,.\end{aligned}$$

The same constructions apply to any other rational parabolic subgroup of SL_2; as they can be thought as the stabilizers of the rationals we will write $\mathbb{P}=\mathbb{P}_\xi$ where $\xi\in\mathbb{Q}\cup\{\infty\}$. We have

$$P_\xi \cong \mathrm{N}_{\mathbb{P}_\xi}\times\mathrm{A}_{\mathbb{P}_\xi}\times\mathrm{M}_{\mathbb{P}_\xi}, \qquad \mathfrak{H}\cong\mathrm{N}_{\mathbb{P}_\xi}\times\mathrm{A}_{\mathbb{P}_\xi}\times\mathrm{X}_{\mathbb{P}_\xi}.$$

The idea one should keep in mind is that the horosperical decomposition allows us to formalize the fact that the rational parabolic subgroups of SL_2 "sit above" the rational numbers at the boundary of the upper-half plane.

Definition 4.1. *The boundary component associated to $\mathbb{P}_\xi$ is defined to be*

$$e(\mathbb{P}_\xi) = N_{\mathbb{P}_\xi}\times X_{\mathbb{P}_\xi}.$$

Remark 4.1. *In our case the boundary component associated to every parabolic subgroup is especially simple, indeed $e(\mathbb{P}_\xi)\cong\mathbb{R}\times\{\xi\}$. Moreover, $e(\mathbb{P}_\xi)$ is a $N_{\mathbb{P}_\xi}$-bundle over $X_{\mathbb{P}_\xi}$ which can be interpreted as the locally symmetric domain for $\mathbb{L}_{\mathbb{P}_\xi}$, the Levi subgroup of $\mathbb{P}_\xi$. More precisely,*

$$e(\mathbb{P}_\xi)\cong P_\xi/A_{\mathbb{P}_\xi}K_{\mathbb{P}_\xi}$$

where $K_{\mathbb{P}_\xi} = P_\xi \cap SO_2(\mathbb{R})$ *and*

$$X_{\mathbb{P}_\xi} = N_{\mathbb{P}_\xi} \backslash P_\xi / A_{\mathbb{P}_\xi} K_{\mathbb{P}_\xi}.$$

Let's see what happens for the parabolic $\mathbb{B}_2$*. It has a Levi decomposition of the following form* $\mathbb{B}_2 = \mathbb{U}_2 \cdot \mathbb{G}_m$ *and the locally symmetric domain for* $\mathbb{G}_m$ *is just the one-point space:*

$$A_{\mathbb{G}_m} \backslash \mathbb{G}_m(\mathbb{R}) / K_\infty = \mathbb{R}_{>0} \backslash \mathbb{R}^\times / \{\pm 1\} = \{pt\}.$$

Definition 4.2 (Bordification)**.** *The Borel-Serre bordification of the upper-half plane is the set*

$$\overline{\mathfrak{H}}^{BS} = \mathfrak{H} \bigsqcup_{\mathbb{P} \text{ rat. parabolic}} e(\mathbb{P}) = \mathfrak{H} \bigsqcup_{\xi \in \mathbb{Q} \cup \{\infty\}} (\mathbb{R} \times \{\xi\}).$$

To formalize the fact that we want to put the boundary components near the rational points at the boundary of the upper-half plane in such a way that they become parameter space for geodesics, we define the topology on the bordification $\overline{\mathfrak{H}}^{\mathrm{BS}}$ using the theory of convergent sequences [5, III.8.13, p.114]. We give $\mathfrak{H}$ and the boundary components $e(\mathbb{P})$ their natural topologies and then we say that a sequence $\{z_j\}_j \subseteq \mathfrak{H}$ converges to a boundary point $(n_\infty, \xi) \in e(\mathbb{P}_\xi)$ if, under the horospherical decomposition of $\mathfrak{H}$ with respect to $\mathbb{P}_\xi$, we can write

$$z_j = (n_j, a_j, \xi_j) \in \mathfrak{H} \cong \mathrm{N}_{\mathbb{P}_\xi} \times \mathrm{A}_{\mathbb{P}_\xi} \times \mathrm{X}_{\mathbb{P}_\xi}$$

and

- $n_j \to n_\infty$ in $N_{\mathbb{P}_\xi}$,
- $\xi_j \to \xi$ in $\mathrm{X}_{\mathbb{P}_\xi} = \{\xi\}$,
- $\chi(a_j) \to +\infty$ in $\mathbb{R}^\times$ where $\chi: \ \mathrm{A}_{\mathbb{P}_\xi} \to \mathbb{G}_m(\mathbb{R})$ is the character through which $\mathrm{A}_{\mathbb{P}_\xi}$ acts on $\mathrm{Lie}(\mathrm{N}_{\mathbb{P}_\xi})$

Let's unravel this definition for $\mathbb{P}_\infty = \mathbb{B}_2$ (which is enough to understand them all because of the $\mathrm{SL}_2(\mathbb{Q})$ transitive action on rational Borel subgroups). In this case, $\mathrm{A}_{\mathbb{P}_\infty}$ acts via the character

$$\chi \begin{pmatrix} \alpha_j & \\ & \alpha_j^{-1} \end{pmatrix} = \alpha_j^2$$

so a sequence $\{z_j\}_j \subset \mathfrak{H}$ converges to some point (x_∞, ∞) of $e(\mathbb{P}_\infty) = \mathbb{R} \times \{\infty\}$ if the imaginary parts diverge $y_j = \mathrm{Im} z_j \to +\infty$ and if the real parts converge to to the given point $x_j \to x_\infty$.

Proposition 4.1. *The* $SL_2(\mathbb{Q})$*-action on* $\mathfrak{H}$ *extends to a continuous action on* $\overline{\mathfrak{H}}^{BS}$*.*

Proof. This is [5, III.9.15, p.333], but let's see how we can extend the action. Given $g \in \mathrm{SL}_2(\mathbb{Q})$ we write $\mathrm{g} = \mathrm{kp} = \mathrm{knam}$ using the decompositions

$$\mathrm{SL}_2(\mathbb{R}) = \mathrm{SO}_2(\mathbb{R}) P_\xi \cong \mathrm{SO}_2(\mathbb{R}) \left(\mathrm{N}_{\mathbb{P}_\xi} \times \mathrm{A}_{\mathbb{P}_\xi} \times \mathrm{M}_{\mathbb{P}_\xi} \right).$$

Then we define

$$\mathrm{g} : e(\mathbb{P}_\xi) \longrightarrow e(\mathbb{P}_{g(\xi)})$$

by $\mathrm{g}(\mathrm{n}', \mathrm{m}') = ({}^{\mathrm{kam}}(\mathrm{nn}'), {}^{\mathrm{k}}(\mathrm{mm}'))$. ∎

The final theoretical result is the following.

Theorem 4.1. *Let $\Gamma \leq SL_2(\mathbb{Q})$ be an arithmetic subgroup, then $\overline{\mathfrak{H}}^{BS}/\Gamma$ is a compact Hausdorff space. Moreover, if Γ is torsion-free, the quotient has a canonical structure of real analytic manifold with corners.*

Proof. [5, III.9.18, p.337] ☐

If we try to picture what is the Borel-Serre compactification $\overline{X_0(N)}^{\mathrm{BS}}$, it's easy to see that we are compactifying the open modular curve by adding an $\mathbb{S}^1$ at every cusp. Indeed,

$$\Gamma_0(N) \cap \mathrm{B}_2 = \left\{ \pm \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

and one can compute that the action of these elements on $e(\mathbb{B}_2) = \mathbb{R}$ is by translation by the upper-right entry of the matrix, therefore

$$(\mathrm{B}_2 \cap \Gamma) \backslash e(\mathbb{B}_2) \cong \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1.$$

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