The Riemann-Hilbert Problem

Gabriel Chênevert

Final project for the course
Vector Bundles on Curves
Prof. Eyal Z. Goren

May 2002

1 Introduction

Consider a linear system of differential equations of one complex variable

$$\frac{dw_i}{dz} = \sum_{j=1}^{n} b_{ij}(z)\, w_j(z), \quad i = 1, \ldots, n \tag{1}$$

having singularities at points $a_1, \ldots, a_m$; that is, suppose that the $b_{ij}(z)$'s are holomorphic functions on $\mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$. Locally, such a system always admit a solution, but global solutions are multi-valued in general. Analytic continuation of a local solution along a closed curve may lead to another solution if the curve encloses some singularities, so that one gets this way a representation of the fundamental group

$$\pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}) \longrightarrow \mathrm{GL}(n, \mathbb{C})$$

on the space of solutions. The Riemann-Hilbert problem asks: can any representation of this fundamental group be obtained this way, if possible by starting with a system of differential equation having "nice" singularities ?

We will see, at least for non-compact Riemann surfaces, that the answer to this question is basically "yes". Then we will hint at how this correspondence between representations of the fundamental group of a Riemann surface and systems of differential equations generalizes to higher dimensional complex analytic manifolds by introducing the concept of flat connections.

2 Linear differential equations

First, we need to recall some results and terminology pertaining to linear differential equations on Riemann surfaces.

For all the following discussion, let X be a Riemann surface. We will denote by $\mathcal{O}$ its structure sheaf, Ω its sheaf of holomorphic 1-forms and $p : \widetilde{X} \to X$ its universal covering.

Definition 2.1 A *linear differential equation* on X is a system of equations in $\Omega(X)$ of the form

$$dw_i = \sum_{j=1}^{n} w_j \, a_{ij}, \quad i = 1, \ldots, n, \tag{2}$$

where the a_{ij}'s are 1-forms, for which we seek solutions $w_1, \ldots, w_n \in \mathcal{O}(X)$.

Example 2.2 The system (1) defines a linear differential equation on $\mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$ in this sense by setting $a_{ij} := b_{ij}(z)dz$.

Example 2.3 A single linear differential equation of the form

$$\frac{d^n y}{dz^n} + b_1(z)\frac{d^{n-1}y}{dz^{n-1}} + \cdots + b_n(z)y = 0,$$

where the $b_i(z)$'s are meromorphic functions on $\mathbb{P}^1$ having poles at points a_1, …, a_m, may be viewed as a linear differential equation on $\mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$ by setting $w_i := y^{(i-1)}$, $i = 1, \ldots, n$, i.e. considering the system

$$\frac{dw_i}{dz} = w_{i+1}, \quad i = 1, \ldots, n-1,$$
$$\frac{dw_n}{dz} = -b_n(z)w_1 - \ldots - b_1(z)w_n.$$

The system of equations (2) may be written more concisely

$$dw = Aw$$

if we set $A := (a_{ij})$ and consider w as the column vector $(w_1, \ldots, w_n)^t$. We now have $A \in \Omega(X)^{n\times n}$ and we seek solutions $w \in \mathcal{O}(X)^n$ for the preceding equality to hold in $\Omega(X)^n$.

If we choose a local chart (U, z), each holomorphic 1-form a_{ij} may be written $a_{ij} = b_{ij}\, dz$ where $b_{ij} \in \mathcal{O}(U)$. So $A = B\, dz$, where $B := (b_{ij}) \in \mathcal{O}(U)^{n\times n}$, and the equation becomes

$$\frac{dw}{dz} = B(z)\, w(z),$$

that is, a system of the form (1) on a open subset of the complex plane.

Locally, for each choice of $c \in \mathbb{C}^n$, such a system admits a unique solution w satisfying $w(0) = c$ ([4], Th. 11.2), which can be analytically continued to a global solution if X is simply connected ([4], Th. 11.4).

In the general case in which X is multiply connected, since we have $p^*A \in \mathcal{O}(\widetilde{X})^{n\times n}$, we get a differential equation

$$dw = (p^*A)w$$

on $\widetilde{X}$ which admits global solutions ([4], Cor. 11.5).

3 Monodromy

So, given $A \in \Omega(X)^{n\times n}$, let $\mathscr{L}_A$ be the n-dimensional (complex) vector space of solutions $w \in \mathcal{O}(\widetilde{X})$ to the equation $dw = (p^*A)w$.

Let $G := \operatorname{Deck}(\widetilde{X}/X)$ be the group of deck transformations of the universal covering of X, which we know is naturally isomorphic to $\pi_1(X)$. Then G acts linearly on the vector space $\mathscr{L}_A$ via

$$\sigma w := (\sigma^{-1})^* w = w \circ \sigma^{-1}.$$

Indeed, if w is a solution of the differential equation then so is σw, because, setting $\tau := \sigma^{-1} \in G$, we have

$$d(\sigma w) = d(\tau^* w) = \tau^* dw = \tau^*(p^* A w) = (p\circ\tau)^* A\, \tau^* w = A\, \sigma w.$$

Thus, we get a representation of $G \simeq \pi_1(X)$ on the space $\mathscr{L}_A$ of solutions.

We note that this representation tells us exactly how local solutions on X of $dw = Aw$ behave when we continue them analytically along a closed curve. Indeed, fix a point $x_0 \in X$ and consider a local solution w of the equation near x_0. Let $\sigma \in \pi_1(X, x_0)$ be the homotopy class of a closed curve u, and $y_0 \in \widetilde{X}$ be some point in the fiber above x_0. Then we may view w as a local solution of $dw = (p^*A)w$ near y_0, and continuing w along u in X corresponds to continuing it in $\widetilde{X}$ along some lifting $\widetilde{u}$ of u starting at y_0. If z_0 is the endpoint of $\widetilde{u}$, then via the standard isomorphism, $\sigma \in \pi_1(X, x_0)$ corresponds to the deck transformation that sends y_0 to z_0, so that $\sigma w = w \circ \sigma^{-1}$ is exactly the local solution near z_0 that we get this way.

Definition 3.1 If $w_1, \ldots, w_n$ constitute a basis for $\mathscr{L}_A$, we call the matrix

$$\Phi := (w_1, \ldots, w_n)$$

a *fundamental system of solutions* for the equation $dw = Aw$.

If Φ is a fundamental system of solutions for $dw = Aw$, then for each $x \in \widetilde{X}$, $\Phi(x)$ is an invertible matrix, so Φ is really an holomorphic mapping

$$\Phi : \widetilde{X} \longrightarrow \mathrm{GL}(n, \mathbb{C}).$$

For short, we will call $\mathrm{GL}(n, \mathcal{O}(\widetilde{X}))$ the subset of $\mathcal{O}(\widetilde{X})^{n\times n}$ consisting of all such mappings, so that we have $\Phi \in \mathrm{GL}(n, \mathcal{O}(\widetilde{X}))$.

We may remark that the set of fundamental systems of the equation $dw = Aw$ is precisely the set of $\Phi \in \mathrm{GL}(n, \mathcal{O}(\widetilde{X}))$ satisfying the matrix equation

$$d\Phi = A\Phi.$$

Also, let us remark that the set $\mathrm{GL}(n, \mathcal{O}(\widetilde{X}))$ is closed under the operation of taking matrix inverses. Indeed, if $\Phi \in \mathrm{GL}(n, \mathcal{O}(\widetilde{X}))$, then $\det\Phi$ is a nowhere vanishing holomorphic function, so that $1/\det\Phi$ is also holomorphic, and hence

$$\Phi^{-1} = \frac{1}{\det\Phi} \operatorname{adj}\Phi \in \mathrm{GL}(n, \mathcal{O}(\widetilde{X})).$$

The action of G on solutions extends to an action on fundamental systems by setting

$$\sigma\Phi := (\sigma w_1, \ldots, \sigma w_n)$$

if $\Phi = (w_1, \ldots, w_n)$.

Since each σw_j is a solution, it may be written as a linear combination of the basis elements $w_1, \ldots, w_n$, i.e.

$$w_j = \sum_{i=1}^{n} t_{ij} w_i.$$

If we set $T_\sigma := (t_{ij}) \in \mathrm{GL}(n, \mathbb{C})$, the matrix that represents the action of σ on $\mathscr{L}_A$ in the basis $w_1, \ldots, w_n$, then we may verify that

$$\sigma\Phi = \Phi T_\sigma.$$

We will refer to this behavior by saying that Φ is *automorphic* with *factors of automorphy* T_σ, $\sigma \in G$.

The function $T : G \to \mathrm{GL}(n, \mathbb{C})$ defined by $T(\sigma) := T_\sigma$ is an homomorphism, because, if $\sigma, \tau \in G$, we have

$$\Phi T_{\tau\sigma} = \tau\sigma\Phi = \tau(\Phi T_\sigma) = (\tau\Phi)(\tau T_\sigma) = \Phi T_\tau T_\sigma,$$

and thus $T_{\tau\sigma} = T_\tau T_\sigma$.

If we choose another fundamental system of solutions Ψ to start with, then there exists $S \in \mathrm{GL}(n, \mathbb{C})$ such that $\Psi = \Phi S$. Then, for $\sigma \in G$, we have

$$\sigma\Psi = \sigma(\Phi S) = \sigma\Phi S = \Phi T_\sigma S = \Psi S^{-1} T_\sigma S.$$

Hence Ψ has factors of automorphy $S^{-1}T_\sigma S$ conjugate to those of Φ, meaning that the homomorphism $T' : G \to \mathrm{GL}(n, \mathbb{C})$ that we get from Ψ is conjugate to T.

Conversely, if $T' : G \to \mathrm{GL}(n, \mathbb{C})$ is another homomorphism conjugate to T, so that there exists $S \in \mathrm{GL}(n, \mathbb{C})$ such that $T'(\sigma) = S^{-1}T_\sigma S$ for all $\sigma \in G$, then it is easy to verify that

$$\Psi := \Phi S \in \mathrm{GL}(n, \mathcal{O}(\widetilde{X}))$$

is a fundamental system of solutions of $dw = Aw$ having its factors of automorphy given by T', since

$$d\Psi = d\Phi S = A\Phi S = A\Psi$$

and

$$\sigma\Psi = \sigma\Phi S = \Phi T_\sigma S = \Psi S^{-1} T_\sigma S = \Psi T'_\sigma.$$

This fact allows us to make the following definition.

Definition 3.2 The well-defined conjugacy class of representations

$$\pi_1(X) \simeq \mathrm{Deck}(\widetilde{X}/X) \longrightarrow \mathrm{GL}(n, \mathbb{C})$$

arising this way from factors of automorphy of fundamental systems of solutions of the differential equation $dw = Aw$ is called the *monodromy* of this equation.

We may now state precisely the Riemann-Hilbert problem: given a representation

$$T : \pi_1(X) \longrightarrow \mathrm{GL}(n, \mathbb{C}),$$

does there exist a linear differential equation on X with monodromy given by T ?

We might also want to impose some more conditions on the kind of system that we seek. For example, if $X := \mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$, we might want to find an equation which has poles of order at most one at $a_1, \ldots, a_m$, which we call a *Fuschian* differential equation.

Remark 3.3 Given a Riemann surface X and a representation $T : \pi_1(X) \to \mathrm{GL}(n, \mathbb{C})$, the problem of finding $A \in \Omega(X)^{n\times n}$ such that the differential equation $dw = Aw$ has monodromy T may be reduced to the problem of finding $\Phi \in \mathrm{GL}(n, \mathcal{O}(\widetilde{X}))$ with factors of automorphy T_σ, $\sigma \in G$.

Indeed, suppose that we have found such a Φ. Then we may set

$$\widetilde{A} := (d\Phi)\Phi^{-1} \in \Omega(\widetilde{X})^{n\times n}.$$

We remark that this matrix is invariant under covering transformations, because for $\sigma \in G$ we have

$$\sigma\widetilde{A} = (\sigma d\Phi)\sigma\Phi^{-1} = d(\sigma\Phi)(\sigma\Phi)^{-1} = d(\Phi T_\sigma)(\Phi T_\sigma)^{-1} = d\Phi\, T_\sigma T_\sigma^{-1}\Phi^{-1} = \widetilde{A}.$$

So to $\widetilde{A} \in \Omega(\widetilde{X})^{n\times n}$ corresponds a matrix $A \in \Omega(X)^{n\times n}$ such that $p^*A = \widetilde{A}$. Then clearly Φ is a fundamental system of solutions for the system $dw = Aw$, so that this system has the required monodromy.

4 The case of the punctured disk

Let us restrict our attention to the punctured disk

$$X := \{z \in \mathbb{C} \,|\, 0 < |z| < R\}, \quad R > 0.$$

Of course, this special case is particularly important, since it will give us information on how the system (1) behaves near one of the singularities a_i.

If we let

$$\widetilde{X} := \exp^{-1}(X) = \{\tilde{z} \in \mathbb{C} \,|\, \Re(\tilde{z}) < \log R\},$$

then we know that

$$p := \exp|_{\widetilde{X}} : \widetilde{X} \longrightarrow X$$

is the universal covering of X, and that the group $\mathrm{Deck}(\widetilde{X}/X)$ of covering transformations is an infinite cyclic group. Let us consider the generator σ of $\mathrm{Deck}(\widetilde{X}/X)$ defined by $\sigma(\tilde{z}) := \tilde{z} - 2\pi i$, so that we have $\sigma\tilde{z} = \tilde{z} + 2\pi i$, where $\tilde{z}$ denotes the usual coordinate on $\widetilde{X} \subseteq \mathbb{C}$.

So let us consider a differential equation $w' = Aw$ with $A \in \mathcal{O}(X)^{n\times n}$ and let $\Phi \in \mathrm{GL}(n, \mathcal{O}(\widetilde{X}))$ be a fundamental system for this equation. Since G is generated by σ, the monodromy of the system is entirely determined by the matrix $T_\sigma \in \mathrm{GL}(n, \mathbb{C})$ satisfying $\sigma\Phi = \Phi T_\sigma$. We call T_σ the *monodromy matrix* associated to Φ.

Recall that the surjective mapping $\exp : \mathbb{C}^{n\times n} \to \mathrm{GL}(n, \mathbb{C})$ defined by

$$\exp A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

extends to a mapping $\exp : \mathcal{O}(X) \to \mathrm{GL}(n, \mathcal{O}(X))$ for any Riemann surface X.

Theorem 4.1 *If X is the punctured disk as above, then for any representation $\pi_1(X) \to \mathrm{GL}(n, \mathbb{C})$ there exists $A \in \mathcal{O}(X)^{n\times n}$ such that the monodromy of the differential equation $w' = Aw$ on X is given by this homomorphism.*

Proof. Let $T \in \mathrm{GL}(n, \mathbb{C})$ be the image of the generator σ of $\mathrm{Deck}(\widetilde{X}/X)$ by the given homomorphism. As was noted earlier, it is sufficient to find $\Phi \in \mathrm{GL}(n, \mathcal{O}(\widetilde{X}))$ such that $\sigma^n\Phi = \Phi T^n$, for all $n \in \mathbb{Z}$. For these equalities to hold, it is only necessary to make sure that $\sigma\Phi = \Phi T$.

Since $\exp : \mathbb{C}^{n\times n} \to \mathrm{GL}(n, \mathbb{C})$ is surjective, it is possible to find $B \in \mathbb{C}^{n\times n}$ such that $\exp(2\pi i B) = T$. Now let

$$\Phi := \exp(B\tilde{z}) \in \mathrm{GL}(n, \mathcal{O}(\widetilde{X})).$$

We may verify that Φ has the required automorphic behavior, since

$$\sigma\Phi = \sigma \exp(B\tilde{z}) = \exp(B\sigma\tilde{z}) = \exp(B\tilde{z} + 2\pi i B) = \exp(B\tilde{z})\exp(2\pi i B) = \Phi T.$$

To find explicitly the equation that we get, one may note that

$$d\Phi = d\exp(B\tilde{z}) = d(B\tilde{z})\exp(B\tilde{z}) = B\exp(B\tilde{z})d\tilde{z} = B\Phi d\tilde{z}$$

so we get $\widetilde{A} := (d\Phi)\Phi^{-1} = Bd\tilde{z} \in \Omega(\widetilde{X})^{n\times n}$, to which corresponds $A := (1/z)Bdz \in \Omega(X)^{n\times n}$, where z stands for the usual coordinate on X.

Thus the differential equation

$$\frac{dw}{dz} = \frac{1}{z}Bw$$

on X has the required monodromy. ∎

This proof in fact allows us to describe in a nice way the solutions of any linear differential equation on the punctured disk.

Theorem 4.2 *Let $A \in \mathcal{O}(X)^{n\times n}$. Then any fundamental system of solutions $\Phi \in \mathrm{GL}(n, \mathcal{O}(\widetilde{X}))$ of the differential equation $w' = Aw$ on the punctured disk X may be written $\Phi = \Psi\Phi_0$, where $\Phi_0 = \exp(B\tilde{z})$ for a constant matrix $B \in \mathbb{C}^{n\times n}$ and Ψ is invariant under covering transformations, i.e. corresponds to an element of* $\mathrm{GL}(n, \mathcal{O}(X))$.

Proof. Let Φ be a fundamental system of solutions of the given equation and let $T \in \mathrm{GL}(n, \mathbb{C})$ be its monodromy matrix. Just as in the proof of the preceding theorem, it is possible to find $B \in \mathrm{GL}(n, \mathbb{C})$ such that $\Phi_0 := \exp(B\tilde{z})$ has the same monodromy matrix T.

Now let $\Psi := \Phi\Phi_0^{-1}$, so that we have $\Phi = \Psi\Phi_0$, and Ψ is invariant under covering transformations because for the generator σ of $\mathrm{Deck}(\widetilde{X}/X)$, we have

$$\sigma\Psi = \sigma(\Phi\Phi_0^{-1}) = (\sigma\Phi)(\sigma\Phi_0)^{-1} = \Phi T T^{-1}\Phi_0 = \Psi.$$

This concludes the proof. ∎

Let us illustrate how Theorem 4.1 works on a concrete example.

Example 4.3 Let $X := \mathbb{C} \setminus \{0\}$ be the punctured plane and consider the homomorphism $\pi_1(X) \to \mathrm{GL}(n, \mathbb{C})$ that sends the generator σ of $\pi_1(X)$ to $T := \begin{pmatrix} i & 1 & 0 \\ 0 & i & 1 \\ 0 & 0 & i \end{pmatrix}$.

It is easy to check that if we set $B := \begin{pmatrix} 1/4 & -1/2\pi & -i/4\pi \\ 0 & 1/4 & -1/2\pi \\ 0 & 0 & 1/4 \end{pmatrix}$, then $\exp(2\pi i B)$ is the required monodromy matrix T.

By calculating $\Phi := \exp(B\tilde{z})$, we find that

$$\Phi = e^{\tilde{z}/4} \begin{pmatrix} 1 & -\tilde{z}/2\pi & \tilde{z}^2/8\pi^2 - i\tilde{z}/4\pi \\ 0 & 1 & -\tilde{z}/2\pi \\ 0 & 0 & 1 \end{pmatrix}$$

so that

$$w_1 := e^{\tilde{z}/4}\,(1,\, 0,\, 0)^t\,,\; w_2 := e^{\tilde{z}/4}\left(-\frac{\tilde{z}}{2\pi},\, 1,\, 0\right)^t,\; w_3 := e^{\tilde{z}/4}\left(\frac{\tilde{z}^2}{8\pi^2} - \frac{i\tilde{z}}{4\pi},\, \frac{\tilde{z}}{2\pi},\, 1\right)^t$$

form a basis of the space of solutions of the equation $w' = Bw$ on $\widetilde{X}$, and we may readily check that they exhibit the required monodromy. For example,

$$\sigma w_2 = e^{(\tilde{z}+2\pi i)/4}\left(-\frac{\tilde{z}+2\pi i}{2\pi},\, 1,\, 0\right) = ie^{\tilde{z}/4}\left(-\frac{\tilde{z}}{2\pi} - i,\, 1,\, 0\right) = w_1 + iw_2.$$

So, as multi-valued functions,

$$z^{1/4}\,(1,\, 0,\, 0)^t\,,\; z^{1/4}\left(-\frac{\log z}{2\pi},\, 1,\, 0\right)^t,\; z^{1/4}\left(\frac{\log^2 z}{8\pi^2} - \frac{i\log z}{4\pi},\, \frac{\log z}{4\pi},\, 1\right)^t$$

form a basis for the space of solutions of the equation $w' = (1/z)Bw$ on the punctured plane X.

5 Solution for non-compact Riemann surfaces

We will now give the solution to the Riemann-Hilbert problem on any non-compact Riemann surface, using the fact that every holomorphic vector bundle on a non-compact Riemann surface is trivial ([4], Th. 30.4).

Theorem 5.1 *If X is a non-compact Riemann surface, then for any homomorphism $T : \pi_1(X) \to \mathrm{GL}(n, \mathbb{C})$ there exists a linear differential equation on X with monodromy T.*

Proof. Once again, we only need to find a $\Phi \in \mathrm{GL}(n, \mathcal{O}(\widetilde{X}))$ which has the T_σ, $\sigma \in \pi_1(X)$, as factors of homomorphy. Let $G := \mathrm{Deck}(\widetilde{X}/X)$ denote the group of covering transformation of X and consider T as an homomorphism $G \to \mathrm{GL}(n, \mathbb{C})$.

Since $p : \widetilde{X} \to X$ is a covering, we know that each point $x \in X$ has an open neighbourhood U such that its pre-image is a disjoint union of sheets homeomorphic to U via p, i.e.

$$p^{-1}(U) = \bigcup_{\lambda \in \Lambda} S_\lambda$$

where each $p\,|_{S_\lambda} : S_\lambda \to U$ is a homeomorphism.

Let us remark that we may, in fact, index the sheets over U by the elements of G, because if we fix an index $\lambda_0 \in \Lambda$, then for each $\lambda \in \Lambda$, there exists exactly one covering transformation $\sigma \in G$ such that $\sigma(S_{\lambda_0}) = S_\lambda$.

So define

$$\varphi : p^{-1}(U) \longrightarrow U \times G$$

by sending $y \in S_\lambda$ to

$$\varphi(y) := (p(y),\, \sigma),$$

where σ is the unique element of G mapping S_{λ_0} onto S_λ. Note that φ depends on the choice of $\lambda_0 \in \Lambda$.

If we endow G with the discrete topology, it is easy to check that φ is in fact an homeomorphism of $p^{-1}(U)$ onto $U \times G$ such that the following diagram commutes :

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U \times G \arrow[dl, "\mathrm{pr}_U"] \\
& U &
\end{tikzcd}$$

Moreover, if $\varphi(y) = (x, \sigma)$, then for each $\tau \in G$ we have $\varphi(\tau y) = (x, \tau\sigma)$ so φ is compatible with the action of G. Such a fiber-preserving homeomorphism compatible with the action of G is called a *G-chart.*

We remark that any G-chart $\varphi : p^{-1}(U) \to U \times G$ may be decomposed as $\varphi = (p, \eta)$, where $\eta : p^{-1}(U) \to G$ satisfies

$$\eta(\tau y) = \tau\eta(y) \quad y \in p^{-1}(U),\, \tau \in G.$$

Now we may cover X by open subsets U_i with G-charts

$$\varphi_i = (p,\, \eta_i) : p^{-1}(U_i) \to U_i \times G.$$

We set $Y_i := p^{-1}(U_i)$ and define $\Psi_i : Y_i \to \mathrm{GL}(n,\, \mathbb{C})$ by

$$\Psi_i(y) := T_{\eta_i(y)^{-1}}.$$

Each Ψ_i is in $\mathrm{GL}(n,\, \mathcal{O}(Y_i))$ since Ψ_i is locally constant.

We now verify that the Ψ_i have the required automorphic behavior on Y_i. Indeed, for each $\sigma \in G$, we have

$$\sigma\Psi_i(y) = \Psi_i(\sigma^{-1}y) = T_{\eta_i(\sigma^{-1}y)^{-1}} = T_{\eta_i(y)^{-1}\sigma} = T_{\eta_i(y)^{-1}}T_\sigma = \Psi_i T_\sigma.$$

Now we define

$$H_{ij} := \Psi_i\Psi_j^{-1} \in \mathrm{GL}(n,\, \mathcal{O}(U_i \cap U_j)).$$

H_{ij} is invariant under covering transformations since for $\sigma \in G$ we have

$$\sigma H_{ij} = \sigma(\Psi_i\Psi_j^{-1}) = (\sigma\Psi_i)(\sigma\Psi_j)^{-1} = \Psi_i T_\sigma T_\sigma^{-1}\Psi_j = H_{ij},$$

so H_{ij} defines an element $F_{ij} \in \mathrm{GL}(n,\, \mathcal{O}(U_i \cap U_j))$ such that $H_{ij} = p^*F_{ij}$.

Moreover,

$$p^*(F_{ij}F_{jk}) = (p^*F_{ij})(p^*F_{jk}) = H_{ij}H_{jk} = H_{ik} = p^*F_{ik},$$

hence we conclude that the F_{ij}'s satisfy the cocyle relation $F_{ij}F_{jk} = F_{ik}$ on $U_i \cap U_j \cap U_k$, so that we may view them as transition functions for a vector bundle on X. Since every vector bundle on X, a non-compact Riemann surface, is trivial, we know that is it possible to factorize $F_{ij} = F_iF_j^{-1}$ on $U_i \cap U_j$, where each $F_i \in \mathrm{GL}(n,\, \mathcal{O}(U_i))$.

The element $H_i := p^*F_i \in \mathrm{GL}(n,\, \mathcal{O}(Y_i))$ is invariant under covering transformations. We now set

$$\Phi_i := H_i^{-1}\Psi_i \in \mathrm{GL}(n,\, \mathcal{O}(Y_i)).$$

Each Φ_i has the required automorphic behavior since

$$\sigma\Phi_i = (\sigma H_i^{-1})(\sigma\Psi_i) = H_i^{-1}\Psi_i T_\sigma = \Phi_i T_\sigma.$$

Moreover, on $U_i \cap U_j$, we have

$$\Phi_i^{-1}\Phi_j = \Psi_i^{-1}H_iH_j^{-1}\Psi_j = \Psi_i^{-1}H_{ij}\Psi_j = \Psi_i^{-1}\Psi_i\Psi_j^{-1}\Psi_j = 1,$$

so that the $\Phi_i \in \mathrm{GL}(n,\, \mathcal{O}(U_i))$ define a global $\Phi \in \mathrm{GL}(n,\, \mathcal{O}(X))$ which has the required automorphic behavior, i.e.

$$\sigma\Phi = \Phi T_\sigma, \quad \sigma \in G.$$

Then, using Remark 3.3, we obtain the required linear differential equation on X. ■

In particular, this theorem applied to $X := \mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$ guarantees us the existence of a differential equation on X having prescribed monodromy, but tells us nothing about the nature of the singularities that this equation may have in $a_1, \dots, a_m$. We will now assess this problem.

Definition 5.2 Let X be a Riemann surface, $S \subset X$ a closed discrete subset of X and $X' := X \setminus S$. Let $A \in \Omega(X')^{n\times n}$. We say that the differential equation $dw = Aw$ on X' has a *regular singular point* or a singularity of *Fuchsian type* at $a \in S$ if each fundamental system of solutions Φ on the universal covering of X' has at most poles of first order at points of $p^{-1}(\{a\})$.

Example 5.3 The linear differential equation

$$\frac{dw}{dz} = A(z)w$$

on $\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$ is Fuchsian if and only if $A(z)$ may be written

$$A(z) = \sum_{i=1}^{m} \frac{1}{z - a_i} A_i,$$

where the $A_i \in \mathbb{C}^{n\times n}$ are constant matrices ([1], 1.2.1).

Theorem 5.4 ([4], Th. 11.13) *If the matrix $A \in \Omega(X')^{n\times n}$ has at most a pole of first order at $a \in S$, then the differential equation $dw = Aw$ on X' has a regular singular point at $a \in S$.*

Theorem 5.5 ([4], Th. 31.5) *Let X be a non-compact Riemann surface and $S \subset X$ a closed discrete subset. Then for each homomorphism $T : \pi_i(X \setminus S) \to \mathrm{GL}(n, \mathbb{C})$, there exists a linear differential equation on $X \setminus S$ with regular singular points at each point of S, which has T as monodromy.*

Proof. Let us write $S = \{a_i \,|\, i \in I\}$. Then for each $i \in I$ we may find a coordinate neighbourhood (U_i, z_i) of a_i containing no other point of S and which is a punctured disk ([4], Lemma 31.4). Let J consist of I plus one special symbol not in I added, say 0, and set $U_0 := X \setminus S$. Now we got an open covering $(U_j)_{j\in J}$ of X. Let $p : Y \to X \setminus S$ the universal covering of $X \setminus S$ and let, for $i \in I$, $Y_i := p^{-1}(U_i \setminus \{a_i\})$. Then $p|_{Y_j} : Y_j \to U_j$ is the universal covering of U_j for each $j \in J$.

By applying Theorem 5.1 to $U_0 = X \setminus S$, we are able to find $\Psi_0 \in \mathrm{GL}(n, \mathcal{O}(Y_0))$ such that $\sigma\Psi_0 = \Psi T_\sigma$ for each $\sigma \in \pi_1(X \setminus S)$.

Also, Theorem 4.1 gives us the existence of $\Psi_i \in \mathrm{GL}(n, \mathcal{O}(Y_i))$ having a regular singular point at $a_i \in U_i$ and which has the same automorphic behavior as Ψ_0 on Y_i ($i \in I$). For $i, j \in I$, set

$$H_{ij} := \Psi_i \Psi_j^{-1} \in \mathrm{GL}(n, \mathcal{O}(Y_i \cap Y_j)).$$

Since H_{ij} is invariant under covering transformations, it determines an element

$$F_{ij} \in \mathrm{GL}(n,\, \mathcal{O}(U_i \cap U_j))$$

such that $H_{ij} = p^* F_{ij}$.

As in the previous proof, we may view the F_{ij}'s as transition functions for a vector bundle on X, and since every vector bundle on X is trivial, there exist $F_i \in \mathrm{GL}(n,\, \mathcal{O}(U_i))$ such that $F_{ij} = F_i F_j^{-1}$ on $U_i \cap U_j$. Now, we set

$$\Phi_j := F_j^{-1} \Psi_j \in \mathrm{GL}(n,\, \mathcal{O}(Y_i)).$$

Just as in the previous proof, we may verify that each Φ_j has the required automorphic behavior, and that they piece together to yield $\Phi \in \mathrm{GL}(n,\, \mathcal{O}(Y))$ such that $\sigma\Phi = \Phi T_\sigma$ for all $\sigma \in \pi_1(X \setminus S)$.

Since on $Y_i = p^{-1}(U_i \setminus \{a_i\})$ we have $\Phi = F_i^{-1}\Psi_i$, then Φ has only regular singular points because this is the case for Ψ_i and that F_i is homolorphic on U_i. Now we set $A := (d\Phi)\Phi^{-1} \in \Omega(Y)^{n\times n}$ which defines an element of $\Omega(X \setminus S)^{n\times n}$ since it is invariant under covering transformations. ■

6 Flat Connections

In this section, we will look at the generalization of linear differential equations on complex analytic manifolds. Let X be a complex analytic manifold, and let $V \to X$ be a holomorphic vector bundle of rank n on X, and $\mathcal{F}$ the locally free sheaf of rank n of holomorphic sections of V.

Moreover, let $\Omega^i := \bigwedge^i \Omega$ be the sheaf of i-forms on X. Set $\Omega^i(\mathcal{F}) := \Omega^i \otimes_{\mathcal{O}} \mathcal{F}$ the sheaf of i-forms with coefficients in $\mathcal{F}$.

Definition 6.1 A *connection* on $\mathcal{F}$ is a $\mathbb{C}$-linear sheaf homomorphism

$$\nabla : \mathcal{F} \longrightarrow \Omega(\mathcal{F}) = \Omega \otimes \mathcal{F}$$

such that for all $f \in \mathcal{F}$, $\varphi \in \mathcal{O}$ we have

$$\nabla(\varphi f) = d\varphi \otimes f + \varphi \nabla f.$$

Definition 6.2 A *morphism* $\varphi : (\mathcal{F},\, \nabla) \to (\mathcal{G},\, \nabla')$ between two flat connections over X is a morphism of $\mathcal{O}$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\nabla"'] & \mathcal{G} \arrow[d, "\nabla'"] \\ \Omega(\mathcal{F}) \arrow[r, "1_\Omega \otimes \varphi"] & \Omega(\mathcal{G}) \end{tikzcd}$$

It is easy to see that locally free sheaves over X endowed with flat connection with connection morphisms form a category.

Example 6.3 If we take the trivial bundle $V := X \times \mathbb{C}^n$, i.e. the free sheaf $\mathcal{F} = \mathcal{O}^n$, then $\Omega(\mathcal{F}) = \Omega \otimes_{\mathcal{O}} \mathcal{O}^n = \Omega^{\oplus n}$, and the usual coordinate-wise exterior derivative

$$d : \mathcal{O}^n \longrightarrow \Omega^{\oplus n}, \quad w \longmapsto dw$$

is a connection on $\mathcal{F}$, called the *canonical connection* on $\mathcal{O}^n$.

Some constructions are available in the category of locally free sheaves with connections.

Example 6.4 If $(\mathcal{F}_1, \nabla_1)$ and $(\mathcal{F}_2, \nabla_2)$ are two locally free sheaves endowed with connections, then we may define in a natural way a connection on $\mathcal{F}_1 \oplus \mathcal{F}_2$:

$$\nabla_1 \oplus \nabla_2 : \mathcal{F}_1 \oplus \mathcal{F}_2 \longrightarrow \Omega(\mathcal{F}_1 \oplus \mathcal{F}_2) = \Omega(\mathcal{F}_1) \oplus \Omega(\mathcal{F}_2).$$

Example 6.5 There is also a natural connection on $\mathcal{F}_1 \otimes \mathcal{F}_2$:

$$\nabla_1 \otimes 1 + 1 \otimes \nabla_2 : \mathcal{F}_1 \otimes \mathcal{F}_2 \longrightarrow \Omega(\mathcal{F}_1 \otimes \mathcal{F}_2).$$

Example 6.6 We may also define a connection ∇ on $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2)$ by

$$(\nabla f)(v) := \nabla_2 f(v) - f(\nabla_1 v).$$

By the universal property of the tensor product, it is easy to check that a connection on $\mathcal{F}$ induces for each p a unique linear map

$$\nabla : \Omega^p(\mathcal{F}) \longrightarrow \Omega^{p+1}(\mathcal{F})$$

satisfying a graded version of Leibniz's rule:

$$\nabla(\omega \otimes f) = d\omega \otimes f + (-1)^p \omega \wedge \nabla f \quad \text{for } \omega \in \Omega^p,\ f \in \mathcal{F}.$$

Definition 6.7 A connection ∇ on $\mathcal{F}$ is called *integrable* or *flat* if the composition

$$\nabla^2 = \nabla \circ \nabla : \mathcal{F} \longrightarrow \Omega^2(\mathcal{F}),$$

called the *curvature* of ∇, vanishes.

Example 6.8 If X is a Riemann surface, then any connection on a locally free sheaf $\mathcal{F}$ on X is flat since $\Omega^2(\mathcal{F}) = \Omega^2 \otimes \mathcal{F} = 0$.

Example 6.9 It is easy to show that if ∇_1 and ∇_2 are flat, so are $\nabla_1 \oplus \nabla_2$ and $\nabla_1 \otimes 1 + 1 \otimes \nabla_2$.

If ∇ is a flat connection on $\mathcal{F}$, it is easy to verify that we get a cochain complex

$$\mathcal{F} \xrightarrow{\nabla} \Omega(\mathcal{F}) \xrightarrow{\nabla} \Omega^2(\mathcal{F}) \xrightarrow{\nabla} \Omega^3(\mathcal{F}) \xrightarrow{\nabla} \dots,$$

called the *de Rham complex* associated to $(\mathcal{F},\, \nabla)$, because for $\omega \in \Omega^p,\ f \in \mathcal{F}$, we have

$$\begin{aligned}
\nabla^2\big(\omega \otimes f\big) &= \nabla\big(d\omega \otimes f + (-1)^p\, \omega \wedge \nabla f\big) \\
&= \nabla\big(d\omega \otimes f\big) + (-1)^p\, \nabla\big(\omega \wedge \nabla f\big) \\
&= d^2 w \otimes f + (-1)^{p+1}\, d\omega \wedge \nabla f + (-1)^p\, d\omega \wedge \nabla f - \omega \wedge \nabla^2 f \\
&= 0.
\end{aligned}$$

Example 6.10 The usual exterior derivative $d : \mathcal{O}^n \to \Omega^{\oplus n}$ is a flat connection on $\mathcal{O}^n$, and the de Rham complex associated to it is just the usual de Rham complex of X.

7 The Riemann-Hilbert correspondence

Now we will see how the notion of connection on a locally free sheaf generalizes to analytic manifolds the notion of linear differential equation on a Riemann surface.

Let $\mathcal{F}$ be a locally free sheaf of rank n on a Riemann surface X and $\nabla : \mathcal{F} \to \Omega(\mathcal{F})$ a (necessarily flat) connection on $\mathcal{F}$. Let $(U,\, z)$ be a coordinate neighbourhood of X on which $\mathcal{F}$ is free.

Then we get

$$\nabla|_U : \mathcal{O}(U)^n \to \Omega(U)^n = \mathcal{O}(U)^n dz.$$

Let $e_1,\, \ldots,\, e_n$ be the canonical $\mathcal{O}(U)$-basis of $\mathcal{O}(U)^n$, and write

$$\nabla e_i = -\sum_{j=1}^{n} a_{ji} e_j dz, \quad a_{ij} \in \mathcal{O}(U).$$

For an arbitrary $w \in \mathcal{O}(U)^n$, write $w = \sum_{i=1}^{n} w_i e_i$. Then we have

$$\nabla w = \sum_{i=1}^{n} \nabla(w_i e_i) = \sum_{i=1}^{n} (dw_i e_i + w_i \nabla e_i)$$

$$= \sum_{i=1}^{n} \left(dw_i e_i - w_i \sum_{j=1}^{n} a_{ji} e_j dz \right) = \sum_{i=1}^{n} \left(\frac{dw_i}{dz} - \sum_{j=1}^{n} a_{ij} w_j \right) e_i dz$$

hence the equation $\nabla w = 0$ is equivalent to the system of equations

$$\frac{dw_i}{dz} = \sum_{j=1}^{n} a_{ij} w_j, \quad i = 1,\, \ldots,\, n,$$

i.e. the linear differential equation $w' = Aw$ on U, where $A = (a_{ij}) \in \mathcal{O}(U)^{n \times n}$ (see [2], III, 2.2.1).

Conversely, let $dw = Aw$ be a linear differential equation on a Riemann surface X, $A \in \mathcal{O}(X)^{n\times n}$. Then we may define a connection $\nabla : \mathcal{O}^n \to \Omega^{\oplus n}$ by the formula

$$\nabla w := \sum_{i=1}^{n} \left(dw_i - \sum_{j=1}^{n} a_{ij} w_j \right) e_i.$$

It defines a connection because for $f \in \mathcal{O}$, we have

$$\begin{aligned}
\nabla(fw) &= \sum_{i=1}^{n} \left(d(fw_i) - \sum_{j=1}^{n} a_{ij} f w_j \right) e_i \\
&= \sum_{i=1}^{n} \left((f dw_i + df w_i) - \sum_{j=1}^{n} a_{ij} f w_j \right) e_i \\
&= f \sum_{i=1}^{n} \left(dw_i - \sum_{j=1}^{n} a_{ij} w_j \right) e_i + \sum_{i=1}^{n} w_i df e_i \\
&= f \nabla w + df w.
\end{aligned}$$

And we remark that $\nabla w = 0$ if and only if $dw = Aw$.

In the light of what has just been said, it is natural, given a locally free sheaf $(\mathcal{F}, \nabla)$ endowed with a connection, to define the sheaf $\mathcal{F}^{\nabla}$ of *horizontal sections* of $\mathcal{F}$

$$\mathcal{F}^{\nabla} := \{ f \in \mathcal{F} \mid \nabla f = 0 \},$$

i.e.

$$\mathcal{F}^{\nabla}(U) := \{ f \in \mathcal{F}(U) \mid \nabla f = 0 \}.$$

The fact that $\mathcal{F}^{\nabla}$ is indeed a sheaf follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d, "\nabla"'] & \mathcal{F}(V) \arrow[d, "\nabla"] \\
\Omega(\mathcal{F})(U) \arrow[r] & \Omega(\mathcal{F})(V)
\end{tikzcd}$$

in which the horizontal arrows are restriction homomorphisms.

The remarks made earlier prove that $\mathcal{F}^{\nabla}$ is locally isomorphic to $\mathbb{C}^n$. Such a sheaf is called a *local system* on X.

Theorem 7.1 ([2], IV, 1.1, [3], Th. 2.17) *The functor $(\mathcal{F}, \nabla) \to \mathcal{F}^{\nabla}$ is an equivalence of categories between flat connections on X and local systems on X.*

Indeed, if $\mathcal{V}$ is a local system on X, we may associate to it the locally free sheaf $\mathcal{F} := \mathcal{O} \otimes_{\mathbb{C}} \mathcal{V}$ endowed with the flat connection $\nabla : \mathcal{F} \to \Omega(\mathcal{F})$ defined by

$$\nabla(f \otimes v) := df \otimes v.$$

On the other hand, the category of finite dimensional representations of $\pi_1(X)$ is equivalent to the category of local systems on X ([5], Rem. 3.9.2 and [3], Cor. 1.4).

So flat connections corresponds to representations of the fundamental group; this is what is called the *Riemann-Hilbert correspondence*.

References

[1] D. V. Anosov and A. A. Bolibruch, *The Riemann-Hilbert Problem*, Aspects of Mathematics, 1994.

[2] A. Borel *et al.*, *Algebraic $\mathcal{D}$-Modules*, Perspectives in Mathematics, Academic Press, 1987.

[3] P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics 163, Springer-Verlag, 1970.

[4] O. Forster, *Lectures on Riemann Surfaces*, Graduate Texts in Mathematics 81, Springer-Verlag, 1981.

[5] F. Kirwan, *An Introduction to Intersection Homology Theory*, Pitman Research Notes in Mathematics Series 187, 1988.