ASSIGNMENT 9 - MATH 251, WINTER 2008

Submit by Wenesday, March 26, 16:00

(1) Let A be a matrix in block form:

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \\ & & \ddots & \\ 0 & 0 & \cdots & A_k \end{pmatrix}.$$

Prove that

$$\Delta_A = \Delta_{A_1}\Delta_{A_2}\cdots\Delta_{A_r},$$

and

$$m_A = \text{lcm}\{m_{A_1}, m_{A_2}, \cdots, m_{A_r}\}.$$

You may use the formula

$$A^b = \begin{pmatrix} A_1^b & 0 & \cdots & 0 \\ 0 & A_2^b & & \\ & & \ddots & \\ 0 & 0 & \cdots & A_k^b \end{pmatrix}$$

for every positive integer b.

(2) Calculate the characteristic and minimal polynomial of the following matrices with real entries. In each case determine the algebraic and geometric multiplicity of each eigenvalue. Decide which matrix is diagonalizable and for that one, say A, find an invertible matrix M such that $M^{-1}AM$ is diagonal.

$$\begin{pmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{pmatrix}$$

(3) For each matrix N in Exercise 1 (considered as a linear transformation T) find the Primary Decomposition, i.e., the factorization of the minimal polynomial, the kernels of the factors, and for each kernel a matrix representation of T.

(4) Let S and T be commuting linear maps from a vector space V to itself, that is $TS = ST$. Let λ be an eigenvalue of T and let E_λ be the corresponding eigenspace.
  (a) Prove that E_λ is S invariant.
  (b) Conclude that if T is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_r$, and therefore

$$V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_r},$$

  we may decompose S as

$$S = S_1 \oplus \cdots \oplus S_r,$$

  where $S_i : E_{\lambda_i} \to E_{\lambda_i}$.
  (c) Assume that both S and T are diagonalizable. Prove now that there exists a basis of V in which both T and S are diagonal.

(d) Apply all that to the linear maps given by the matrices

$$A = \begin{pmatrix} 3 & -2 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & -2 & 5 \\ 1 & 0 & 5 \\ 0 & 0 & 2 \end{pmatrix}.$$

(5) The following explains how to calculate the minimal polynomial without factoring the characteristic polynomial.

Let $T : V \to V$ be a linear map on a finite dimensional vector space V of dimension n.

(a) Let $\{U_1, U_2, \dots, U_r\}$ be a collection of T-invariant subspaces of V such that $V = U_1 + \cdots + U_r$. Let f_i be the minimal of $T|_{U_i}$. Prove that

$$m_T = lcm(f_1, \dots, f_r).$$

(b) Let v be a non-zero vector in V. Consider the vectors

$$v, Tv, T^2v, T^3v, \dots$$

Suppose that a is the first power of T such that $T^a v$ depends linearly on $v, Tv, \dots, T^{a-1}v$, say $T^a v + b_{a-1}T^{a-1}v + \cdots + b_1 Tv + b_0 v = 0$. Prove that the minimal polynomial of T on $\mathrm{Span}(\{v, Tv, T^2v, \dots\})$ is precisely $t^a + b_{a-1}t^{a-1} + \cdots + b_1 t + b_0$.

(c) A subspace of the form $\mathrm{Span}(\{v, Tv, T^2v, \dots\})$ is called a cyclic subspace. Prove that V is the sum (not necessarily direct sum!) of finitely many cyclic subspaces.

(d) Conclude that one can calculate the minimal polynomial without factoring.

(e) Do that in practice for the matrix

$$A = \begin{pmatrix} 3 & -2 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(6) Prove that the matrix of complex numbers

$$A = \begin{pmatrix} 1 & 1 & 5 & 0 & 0 \\ 7 & 1 & 3 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

is diagonalizable. Note: you don't need to diagonalize it to prove it !!