Submit by Monday, March 17, 16:00

(1) Let $T : V \to V$ be a linear transformation. Let $W$ be a subspace of $V$ such that $T(W) \subseteq W$ (such a subspace is called a $T$-invariant subspace).
   (a) Prove that there is a well defined linear map $T'$ induced by $T$, $T' : V/W \to V/W$.
   (b) Let $A \in M_n(\mathbb{C})$. Prove that there is a matrix $M$ such that $B = M^{-1}AM$ is an upper triangular matrix, i.e. $b_{ij} = 0$ if $i > j$. Suggestion: Argue by induction on the dimension making use of part (a).

(2) Given a monic polynomial of degree $n$, $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$, show that there exists an $n \times n$ matrix $A$, such that $\Delta_A(t) = f(t)$.
   (Suggestion: Look at past assignments.)

(3) Find an explicit formula for $a_n$ in the series $a_0, a_1, a_2, \ldots$ given by
   
   
   
   
   
   
   (4) **Markov Processes.** Imagine a particle that can be in any one of $n$ states $S_1, S_2, \ldots, S_n$.

   (These may be the spin states of a particle, or the room my kid is in...) Initially, we might not know exactly the state the particle is in and have at our disposal only the probability it is in a certain state, that is, we have a probability distribution $f_1, f_2, \ldots, f_n$, $f_i \geq 0$, $f_1 + f_2 + \cdots + f_n = 1$.

   A certain process is now taking place (it is an example of certain stochastic processes called Markov chains). At every time $t = 1, 2, 3, \ldots$ the particle may change its state. We know the probability it change its state from state $S_j$ to state $S_i$ and construct a matrix $M = (m_{ij})$, where $m_{ij}$ is the probability that the particle goes from state $S_j$ to state $S_i$.

   (a) Show that $M$ is a matrix of non-negative real numbers such that for every $j$ we have $\sum_{i=1}^{n} m_{ij} = 1$.
   (b) Prove that the probability distribution describing the state of the particle at time $n$ is given by

   $M^n \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$.

   (c) Consider the case of two states $S_1, S_2$ where $M$ (called the transition matrix) is given by

   $ \begin{pmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{pmatrix}$.

   Show that for any initial probability distribution $(f_1, f_2)$ the limit

   $\lim_{n \to \infty} M^n \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

   exists, and compute it.

(5) Find the characteristic polynomial, eigenvalues $\lambda$ and bases for the spaces $E_\lambda$ for the following matrices:

   $ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $ \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$,

   over the fields: (i) $\mathbb{R}$, (ii) $\mathbb{C}$, (iii) $\mathbb{Z}/2\mathbb{Z}$, (iv) $\mathbb{Z}/5\mathbb{Z}$. 