ASSIGNMENT 8 - MATH 251, WINTER 2008

Submit by Monday, March 17, 16:00

(1) Let $T : V \to V$ be a linear transformation. Let W be a subspace of V such that $T(W) \subseteq W$ (such a subspace is called a T-invariant subspace).
  (a) Prove that there is a well defined linear map T' induced by T, $T' : V/W \to V/W$.
  (b) Let $A \in M_n(\mathbb{C})$. Prove that there is a matrix M such that $B = M^{-1}AM$ is an upper triangular matrix, i.e. $b_{ij} = 0$ if $i > j$. Suggestion: Argue by induction on the dimension making use of part (a).

(2) Given a monic polynomial of degree n, $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$, show that there exists an $n \times n$ matrix A, such that
$$\Delta_A(t) = f(t).$$
(Suggestion: Look at past assignments.)

(3) Find an explicit formula for a_n in the series $a_0, a_1, a_2, \dots$ given by
$$0,\ 1,\ 1,\ 3,\ 5,\ 11,\ 21,\ 43,\ \dots.$$

(4) **Markov Processes.** Imagine a particle that can be in any one of n states
$$S_1, S_2, \dots, S_n.$$
(These may be the spin states of a particle, or the room my kid is in...) Initially, we might not know exactly the state the particle is in and have at our disposal only the probability it is in a certain state, that is, we have a *probability distribution*
$$(f_1, f_2, \dots, f_n), \quad f_i \geq 0, \quad f_1 + f_2 + \cdots + f_n = 1.$$
A certain process is now taking place (it is an example of certain stochastic processes called *Markov chains*). At every time $t = 1, 2, 3, \dots$ the particle may change its state. We know the probability it change its state from state S_j to state S_i and construct a matrix $M = (m_{ij})$, where m_{ij} is the probability that the particle goes from state S_j to state S_i.
  (a) Show that M is a matrix of non-negative real numbers such that for every j we have $\sum_{i=1}^{n} m_{ij} = 1$.
  (b) Prove that the probability distribution describing the state of the particle at time n is given by
$$M^n \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$
  (c) Consider the case of two states S_1, S_2 where M (called the *transition matrix*) is given by
$$\begin{pmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{pmatrix}.$$
  Show that for any initial probability distribution (f_1, f_2) the limit
$$\lim_{n\to\infty} M^n \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$
  exists, and compute it.

(5) Find the characteristic polynomial, eigenvalues λ and bases for the spaces E_λ for the following matrices:
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix},$$
over the fields: (i) $\mathbb{R}$, (ii) $\mathbb{C}$, (iii) $\mathbb{Z}/2\mathbb{Z}$, (iv) $\mathbb{Z}/5\mathbb{Z}$.