ASSIGNMENT 7 - MATH 251, WINTER 2008

Submit by Monday, March 10, 16:00

(1) Use Proposition 7.2.9 to deduce Proposition 7.2.6 in the notes.

(2) Find an orthonormal basis for $\mathbb{C}^2$ with the inner product defined by the matrix $\left(\begin{smallmatrix} 1 & 1+i \\ 1-i & 5 \end{smallmatrix}\right)$.

(3) Perform the Gram-Schmidt process for the basis $\{1, x, x^2\}$ to $\mathbb{R}[x]_2$ with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx.$$

(4) **Least squares approximation.** Consider an experiment whose results are given by a series of points:

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),$$

where $x_1 < x_2 < \cdots < x_n$ and the x_i, y_i are real numbers.

We assume that the actual law governing this data is linear. Namely, that there is an equation of the form $f_{A,B}(x) = Ax + B$ that fits the data up to experimental errors. Therefore, we look for such an equation $Ax + B$ that fits the data best. Our measure for that is "the method of least squares". Namely, given a line $Ax + B$, let $d_i = |y_i - (Ax_i + B)|$ (the distance between the theoretical y and the observed y). Then we seek to minimize

$$d_1^2 + d_2^2 + \cdots + d_n^2.$$

Let

$$T : \mathbb{R}^2 \to \mathbb{R}^n,$$

be the map

$$T(A, B) = (f_{A,B}(x_1), \ldots, f_{A,B}(x_n)).$$

Prove that T is a linear map and that the problem we seek to solve is to minimize

$$\|T(A, B) - (y_1, \ldots, y_n)\|^2.$$

Let W be the subspace of $\mathbb{R}^n$ which is the image of T. Prove that W is two dimensional and that $\{s_1, s_2\}$ is a basis for W, where $s_1 = (1, 1, \ldots, 1), s_2 = (x_1, x_2, \ldots, x_n)$.

Assume for simplicity that $\sum_{i=1}^n x_i = 0$ (this can always be achieved by shifting the data). Find an orthonormal basis for W and use it to find the vector in W closest to $(y_1, \ldots, y_n)$.

Put now everything together to get explicit formulas for A, B such that $f_{A,B}(x)$ is the best linear approximation to the data

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n).$$

(Still under the assumption $\sum_{i=1}^n x_i = 0$.)

(5) **Angles.** Let V be an inner product space over $\mathbb{R}$. Define the angle θ between two non-zero vectors u, v in V to be the unique angle $\theta, 0 \leq \theta \leq \pi$ such that

$$\cos(\theta) = \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|}.$$

This is well-defined by Cauchy-Schwartz. Prove the Law of cosines holds *with this definition* of $\cos(\theta)$: That is, consider a triangle with sides A, B, C of lengths a, b, c, respectively, and let θ be the angle between A and B in the sense defined above. Then

$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

Deduce from the fact that the law of cosines holds in plane geometry for angles defined in the usual way, that our definition of an angle *generalizes* the usual definition.

(6) Let W be the subspace of $\mathbb{F}^4$ defined by the equation $x_1 + x_2 + x_3 + x_4 = 0$. Find the orthogonal projection of $(1, 0, 0, 0)$ on W.