(1) Use Proposition 7.2.9 to deduce Proposition 7.2.6 in the notes.

(2) Find an orthonormal basis for \( \mathbb{C}^2 \) with the inner product defined by the matrix \( \left( \begin{smallmatrix} 1 & 1+i \\ -1-i & 5 \end{smallmatrix} \right) \).

(3) Perform the Gram-Schmidt process for the basis \( \{1, x, x^2\} \) to \( \mathbb{R}[x]_2 \) with respect to the inner product
\[
(f, g) = \int_{-1}^{1} f(x)g(x)dx.
\]

(4) **Least squares approximation.** Consider an experiment whose results are given by a series of points:
\[
(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),
\]
where \( x_1 < x_2 < \cdots < x_n \) and the \( x_i, y_i \) are real numbers.

We assume that the actual law governing this data is linear. Namely, that there is an equation of the form \( f_{A,B}(x) = Ax + B \) that fits the data up to experimental errors. Therefore, we look for such an equation \( Ax + B \) that fits the data best. Our measure for that is “the method of least squares”. Namely, given a line \( Ax + B \), let \( d_i = |y_i - (Ax_i + B)| \) (the distance between the theoretical \( y \) and the observed \( y \)). Then we seek to minimize
\[
d_1^2 + d_2^2 + \cdots + d_n^2.
\]

Let
\[
T : \mathbb{R}^2 \to \mathbb{R}^n,
\]
be the map
\[
T(A, B) = (f_{A,B}(x_1), \ldots, f_{A,B}(x_n)).
\]
Prove that \( T \) is a linear map and that the problem we seek to solve is to minimize
\[
\|T(A, B) - (y_1, \ldots, y_n)\|^2.
\]

Let \( W \) be the subspace of \( \mathbb{R}^n \) which is the image of \( T \). Prove that \( W \) is two dimensional and that \( \{s_1, s_2\} \) is a basis for \( W \), where \( s_1 = (1, 1, \ldots, 1), s_2 = (x_1, x_2, \ldots, x_n) \).

Assume for simplicity that \( \sum_{i=1}^{n} x_i = 0 \) (this can always be achieved by shifting the data). Find an orthonormal basis for \( W \) and use it to find the vector in \( W \) closest to \( (y_1, \ldots, y_n) \).

Put now everything together to get explicit formulas for \( A, B \) such that \( f_{A,B}(x) \) is the best linear approximation to the data
\[
(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n).
\]
(Still under the assumption \( \sum_{i=1}^{n} x_i = 0 \).)

(5) **Angles.** Let \( V \) be an inner product space over \( \mathbb{R} \). Define the angle \( \theta \) between two non-zero vectors \( u, v \) in \( V \) to be the unique angle \( \theta, 0 \leq \theta \leq \pi \) such that
\[
\cos(\theta) = \frac{(u, v)}{\|u\| \cdot \|v\|}.
\]
This is well-defined by Cauchy-Schwartz. Prove the Law of cosines holds with this definition of \( \cos(\theta) \): That is, consider a triangle with sides \( A, B, C \) of lengths \( a, b, c \), respectively, and let \( \theta \) be the angle between \( A \) and \( B \) in the sense defined above. Then
\[
c^2 = a^2 + b^2 - 2ab \cos(\theta).
\]
Deduce from the fact that the law of cosines holds in plane **geometry** for angles defined in the usual way, that our definition of an angle **generalizes** the usual definition.

(6) Let \( W \) be the subspace of \( \mathbb{F}^4 \) defined by the equation \( x_1 + x_2 + x_3 + x_4 = 0 \). Find the orthogonal projection of \((1,0,0,0)\) on \( W \).