ASSIGNMENT 4 - MATH 251, WINTER 2007

Submit by Monday, February 11, 12:00

1. Deduce from the theorems on determinants the following:
   (1) If a column is zero, the determinant is zero.
   (2) $\det(A) = \det(A^t)$, where A^t is the transposed matrix.
   (3) If a row is zero, the determinant is zero.
   (4) $\det(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = \det(v_1, \dots, v_i, \dots, v_j + \alpha v_i, \dots, v_n)$ if $i \neq j$ and $\alpha \in \mathbb{F}$ is any scalar. (This, combined with the fact that "all rules hold for rows too" is very useful to compute determinants by hand, because one can perform row and columns manipulations. For example $\det \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 2 \\ 7 & 8 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 5 \\ 0 & 0 & -8 \\ 0 & -6 & -34 \end{pmatrix}$ (we subtracted from the 2nd and 3rd rows multiples of the first row). This is now equal to $-\det \begin{pmatrix} 1 & 2 & 5 \\ 0 & -6 & -34 \\ 0 & 0 & -8 \end{pmatrix} = -48$.)
   (5) Let A be a matrix in "upper diagonal block form":
   $$A = \begin{pmatrix} A_1 & & & \star \\ 0 & A_2 & & \\ & & \ddots & \\ 0 & & 0 & A_k \end{pmatrix}.$$
   Here each A_i is a square matrix say of size r_i, and A_2 starts at the $r_1 + 1$ column and $r_1 + 1$ row, etc. Prove that
   $$\det(A) = \det(A_1)\det(A_2)\cdots\det(A_k).$$
   Conclude that the determinant of an upper triangular matrix is given by
   $$\det \begin{pmatrix} a_{11} & & & \star \\ 0 & a_{22} & & \\ & & \ddots & \\ 0 & & 0 & a_{kk} \end{pmatrix} = a_{11}a_{22}\cdots a_{kk}.$$
   (Here each a_{ii} is a scalar).
2. Calculate the following series of determinants.
   (1) $\det \begin{pmatrix} 1 & 2 & 2 & 1 \\ 8 & 0 & -1 & 0 \\ 0 & 1 & 1 & 3 \\ 2 & 3 & 1 & 5 \end{pmatrix}$. (Suggestion: try and practice row and column operations to simplify the calculation).
   (2) $\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\det \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \cdots$
   (3) $\det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\det \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $\det \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \cdots$
   (4) $\det \begin{pmatrix} x & -a_2 \\ 1 & x+a_1 \end{pmatrix}$, $\det \begin{pmatrix} x & 0 & a_3 \\ 1 & x & -a_2 \\ 0 & 1 & x+a_1 \end{pmatrix}$, $\det \begin{pmatrix} x & 0 & 0 & -a_4 \\ 1 & x & 0 & a_3 \\ 0 & 1 & x & -a_2 \\ 0 & 0 & 1 & x+a_1 \end{pmatrix}, \cdots$

3. Prove the following formula (the *Vandermonde determinant*):

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots x_1^{n-1} & \\ \vdots & & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots x_n^{n-1} & \end{pmatrix} = \prod_{i>j} (x_i - x_j)$$

For example, for $n = 2, 3$ we have

$$\det \left(\begin{smallmatrix} 1 & x_1 \\ 1 & x_2 \end{smallmatrix}\right) = (x_2 - x_1), \quad \det \left(\begin{smallmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{smallmatrix}\right) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$