ASSIGNMENT 10 - MATH 251, WINTER 2007

Submit by Wednesday, April 9, 12:00, Answer questions 2, 3, 4, 5, 7, 8, 9 for full marks and the additional questions for bonus. It is unlikely the assignment will be marked before the final; you may want to leave yourself a copy.

(1) Let A be the adjacency matrix of a k-regular graph G.
   (a) Prove that k is an eigenvalue of A and find the eigenvector.
   (b) Prove that every eigenvalue λ of A is real and satisfies $|\lambda| \leq k$.
   (c) Prove that if G is not connected, the geometric multiplicity of the eigenvalue k is greater than 1. (The converse is also true.)
   (d) Assume that G is connected. Prove that if G bipartite then $-k$ is an eigenvalue. (The converse is also true.)

(2) Determine the possibilities for the Jordan canonical form of a matrix A with characteristic polynomial $\Delta_A(t) = (t-1)^6(t-2)^4(t-4)^5$ and minimal polynomial $m_A(t) = (t-1)^3(t-2)^2(t-4)$. What if you are given that $\text{null}(A^2 - 3 \cdot A + 2 \cdot I) = 4$?

(3) Find the Jordan canonical form (including the change of basis matrix) of the matrix

$$A = \begin{pmatrix} 5 & 9 & -2 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

(4) Find a formula for the general term of the sequence

$$0, 1, 4, 12, 32, 80, \ldots$$

$(a_{n+2} = 4a_{n+1} - 4a_n)$.

(5) Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map and $\alpha \in \mathbb{C}$ a scalar such that $(T - \alpha Id)^n = Id$. Prove that T is diagonalizable.

(6) It is known that a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ has a
   - **maximum** at a point P if $\partial f/\partial x = \partial f/\partial y = 0$ at P and the 2×2 symmetric matrix

$$-\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

   is positive definite;
   - **minimum** at a point P if $\partial f/\partial x = \partial f/\partial y = 0$ at P and the 2×2 symmetric matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

   is positive definite;
   - **saddle point** at P if the 2×2 symmetric matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

   has one negative eigenvalue and one positive eigenvalue.

If P is either a maximum, minimum or saddle point, we call it a simple critical point. Determine the nature of the simple critical point of the following functions at the origin $(0,0)$

$$f(x,y) = 2x^2 + 6xy + y^2, \quad f(x,y) = x\sin(x) - \cos(y) - xy.$$

(You may view the graphs and rotate them in Maple using
plot3d(2*xˆ 2+6*x*y+yˆ 2, x= -4..4, y = -4..4);
plot3d(x*sin(x) - cos(y) -x*y, x= -4..4, y = -4..4, numpoints=3000);).

Remark. This criterion can be generalized to functions $f : \mathbb{R}^n \to \mathbb{R}$. If all the first partials vanish at a point P and the matrix of mixed derivatives is positive definite (resp. negative definite) at P, then the function has a minimum (resp. maximum) at P.

(7) Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal, where

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

(8) Prove that the determinant of a unitary matrix has absolute value 1.

(9) Prove that the following are equivalent for a matrix $A \in M_n(\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$):
  (a) A is unitary.
  (b) A preserves inner products: $\langle v, w\rangle = \langle Av, Aw\rangle$ for all $v, w \in \mathbb{F}^n$;
  (c) A preserves lengths: $\|Av\| = \|v\|$ for all $v \in \mathbb{C}^n$.

(10) Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a normal operator. Prove the following:
  (a) T is self-adjoint if and only its eigenvalues are real;
  (b) T is unitary if and only if its eigenvalues have absolute value one;
  (c) T is a product US where U is unitary, S is self-adjoint and $US = SU$.

(11) Let A be a real symmetric matrix, $A = (a_{ij})$ of size n. For every $1 \le r \le i$ let $A_r = (a_{ij})_{1\le i,j\le r}$, be the upper left corner $r \times r$ submatrix of A. Prove that A is positive definite if and only if $\det(A_i) > 0$ for $i = 1, 2, \dots, n$.

(For example, the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 8 \end{pmatrix}$ is positive definite because $\det(1) = 1 > 0, \det\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = 1 > 0$ and $\det\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 6 \end{pmatrix} = 3 > 0$.)