ASSIGNMENT 1 - MATH 251, WINTER 2008

Submit by 16:00, Monday, January 21

(1) The following are vector spaces (verify that to yourself). Determine in each case if they are finite dimensional or infinite dimensional by either providing an infinite independent set, or finding a finite basis.

(a) Let S be a non-empty set and $V = \{f : S \to \mathbb{R}\}$ the space of all $\mathbb{R}$-valued functions on S, where we define for $f, g \in S, \alpha \in \mathbb{R}$, the functions $f + g$ and αf using the usual conventions:

$$(f+g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha f(x), \qquad \forall x \in S.$$

(The answer in this case depends on S; distinguish two cases!)

(b) Let $n \geq 0$ an integer. Let $A_0, \dots, A_n$ be scalars (elements of a field $\mathbb{F}$) and let V be the set of infinite vectors $(x_0, x_1, x_2, \dots)$, with coordinates $x_i \in \mathbb{F}$ that satisfy the recursion relation:

$$x_{m+1} = A_n x_m + A_{n-1} x_{m-1} + \cdots + A_0 x_{m-n},$$

for every $m \geq n$. (For example: for $n = 1$ these are the series satisfying $x_2 = A_1 x_1 + A_0 x_0$, $x_3 = A_1 x_2 + A_0 x_1$, $x_4 = A_1 x_3 + A_0 x_2$, etc.. In general, we may also express the relation by

$$\begin{pmatrix} x_{m-n+1} \\ x_{m-n+2} \\ \vdots \\ x_{m+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ A_0 & A_1 & A_2 & \dots & A_n \end{pmatrix} \begin{pmatrix} x_{m-n} \\ x_{m-n+1} \\ \vdots \\ x_m \end{pmatrix}$$

for all $m \geq n$.)

(c) Let $\mathbb{F}$ be a field and V the vector space of all polynomials (of any degree) with coefficients in $\mathbb{F}$.

(2) Prove directly that if S is an independent spanning set then S is a minimal spanning set.

(3) Consider in $\mathbb{R}^4$ the span W of the following set

$$S = \{(1, -1, 1, -1), (1, 3, 2, 2)\}.$$

Describe W as the set of solutions for two linear equations.

(4) Let V be an n-dimensional vector space over a field $\mathbb{F}$. Let $T = \{t_1, \dots, t_m\} \subset V$ be a linearly independent set. Let $W = \mathrm{Span}(T)$. Prove:

$$\dim(W) = m.$$

(5) Let V_1, V_2 be finite dimensional vector spaces over a field $\mathbb{F}$. Prove that

$$\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2).$$

(6) Prove that the set $S = \{(1, 3, 2, 0, 1), (2, 3, 2, 4, 5), (1, -1, 0, 0, 0)\}$ is a linearly independent set in $\mathbb{R}^5$. Use the proof of Steinitz's lemma to find two vectors e_i, e_j, among the standard basis $\{e_1, \dots, e_5\}$ such that $S \cup \{e_i, e_j\}$ is a basis for $\mathbb{R}^5$.

(7) Let $\mathbb{F}$ be a finite field with q elements.

(a) Show that the kernel of the ring homomorphism

$$\mathbb{Z} \to \mathbb{F}$$

defined by $n \mapsto n \cdot 1 = 1 + \cdots + 1$ (n times) is of the form $p\mathbb{Z}$ for some prime p. Conclude that we may assume that $\mathbb{F} \supseteq \mathbb{Z}/p\mathbb{Z}$ for some prime p.

(b) Prove that $\mathbb{F}$ is a vector space of finite dimension over $\mathbb{Z}/p\mathbb{Z}$ and if this dimension is n then $\mathbb{F}$ has p^n elements, and therefore that every finite field has cardinality p^n for some prime p.