February 21, 2008

Quiz 2, MATH 251, Winter 2008

Time: 16:00 - 17:30.

PART I (35% of grade): multiple choice questions. Answer in the exam book. Choose one answer for each question (there is only one correct answer). Do NOT write any explanations (they will be ignored).

(1) What is the determinant of the matrix?

$$\begin{pmatrix} 2 & 5 & 0 & 3 & 4 \\ 1 & 4 & 0 & 2 & 4 \\ 2 & 3 & 1 & 1 & 2 \\ 1 & 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \end{pmatrix}$$

(a) 0.
(b) -1.
(c) -4
(d) $7/2$.

Proof. The determinant is -4. The easiest is to arrange the columns as $3, 4, 5, 1, 2$, then develop using first column. The subdeterminant has the form $\det(A) \cdot \det(B)$. □

(2) Find the rank of the following matrix with integer entries:

$$\begin{pmatrix} 2 & 5 & 0 & 3 & 0 & 4 & 1 & 2 \\ 0 & 3 & 0 & 2 & 1 & -1 & 0 & 1 \\ 0 & -3 & 0 & -2 & -1 & 1 & 0 & -1 \\ 1 & 3 & 1 & 3 & 0 & 2 & 0 & 5 \end{pmatrix}$$

(a) 1.
(b) 2.
(c) 3.
(d) 4.

Proof. The rank is 3. It is easier to find the column rank. We see that we have the vectors $e_1, e_4, e_2 - e_3$ and so the column rank is at least 3. On the other hand every column $\left(\begin{smallmatrix} a \\ b \\ c \\ d \end{smallmatrix}\right)$ satisfies $b + c = 0$ and so the column space is at most 3. □

(3) Find the <u>number</u> of solutions to the following system of linear equations over $\mathbb{Z}/3\mathbb{Z}$.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ x_1 - x_3 &= 1 \\ 2x_1 + 2x_2 + x_3 &= 1 \end{aligned}$$

(a) 3.
(b) 1.
(c) 4.
(d) 9.

Proof. The correct answer is 3. The possible answers imply that the system has at least one solution x. All the solutions are then $x+W$, where W is the solutions to the homogenous system. Row reducing we find the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

whence $\dim(W) = 1$ and so $\sharp W = 3$. ☐

(4) Which of the following lists is a list of matrices that are all row equivalent? (The entries are integers.)

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 2 & 0 & 2 \\ 1 & 1 & 4 & 0 & 7 \\ -1 & 0 & -2 & 1 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 1 & 6 & 0 & 9 \\ 2 & 2 & 8 & 0 & 14 \\ 0 & -5 & -10 & 1 & -26 \end{pmatrix}.$$

(a) A, B, C, D.
(b) A, D.
(c) B, C, D.
(d) A, C, D.

Proof. The matrices A and B are in REF and are not equal, hence not row-equivalent. On the other hand, D row reduces to B. The only list among those given that agrees with this data is B, C, D. ☐

(5) Let $v_1 = (1,0,3,0)$, $v_2 = (2,0,1,0)$ be vectors in $\mathbb{R}^2$. Let x be a variable and

$$v_3 = (0,x,0,5), \ v_4 = (1,x,0,2x).$$

For what values of x is the set $\{v_1, v_2, v_3, v_4\}$ *not* a basis of $\mathbb{R}^4$? (Suggestion: use determinants).
(a) 0, 5/2.
(b) 0.
(c) 0, 5, 2.
(d) 0, 2.

Proof. The correct answer is $0, 5/2$. One uses that a set of 4 vectors forms a basis iff the determinant of the matrix composed of these vectors is non-zero. In our case

$$\det \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & x & x \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 5 & 2x \end{pmatrix} = -\det \begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & x & x \\ 0 & 0 & 5 & 2x \end{pmatrix} = 5x(2x-5).$$

It vanishes for $x = 0, 5/2$. ☐

PART II. (65% of grade)

(1) Let W be the two dimensional subspace of $\mathbb{R}^4$ defined as the solutions to the following system of equations:

$$x_1 - x_2 = 0, \quad x_3 - x_4 = 0.$$

(a) Find a basis for W.

Proof. The rank of the system is 2, thus $\dim(W) = 4-2 = 2$. The vectors $(1,1,0,0), (0,0,1,1)$ are independent and in W, thus a basis. $\square$

(b) Let $U := \text{Span}\{(1,0,0,0),(0,0,1,0)\}$. Prove that

$$\mathbb{R}^4 = W \oplus U$$

(internal direct sum).

Proof. We have $\dim(U) + \dim(W) = 4$ and so, by one of the equivalent characterizations of direct sum, it is enough to prove that $U \cap W = \{0\}$. Now, any vector in U has the shape $(x,0,y,0)$ for some $x, y \in \mathbb{R}$. If that vector belongs to W then $x - 0 = 0, y - 0 = 0$ and so $x = y = 0$. $\square$

(c) Let T be the projection on W along U. Find a matrix A representing T in the usual basis of $\mathbb{R}^4$. Make sure you explain your reasoning. (You may provide the answer in the following form $A = MBM^{-1}$, provided you write down explicitly M, B and M^{-1}. I don't insist that you actually perform the multiplication of matrices.)

Proof. We know that the columns of the matrix $M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ form a basis B for $\mathbb{R}^4$ and $M = {}_{St}M_B$. In the basis B the projection T is given by

$$[T]_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The inverse of M, which is also ${}_BM_{St}$, is given by the matrix

$$M^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

It follows that

$$[T]_{St} = M[T]_B M^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\square$

(2) Let A be an $m \times n$ matrix. Prove that the row rank of A is equal to the column rank of A:

$$\text{rank}_r(A) = \text{rank}_c(A).$$

Remark: we have seen more than one proof of this result. Each proof typically uses a key lemma/theorem, based on which the proof is not hard and consists of a paragraph or two. You may use that key lemma/theorem without proof, but you have to state it precisely.