January 31, 2008

# Quiz 1, MATH 251, Winter 2008

Time: 16:00 - 17:30.

PART I (30% of grade): multiple choice questions. Answer in the exam book. Choose one answer for each question (there is only one correct answer). Do NOT write any explanations (they will be ignored).

(1) Let $V$ be the vector space of infinite series $(a_0, a_1, a_2, \dots)$ of real numbers such that for every $n \geq 2$ we have

$$a_n = a_{n-1} + a_{n-2}.$$

Then:
(a) $V$ has dimension 1.
(b) $V$ has dimension 2. $\Leftarrow\Leftarrow\Leftarrow$
(c) $V$ is infinite dimensional.

*Proof.* We had a question like that on the assignments. The idea is that such a series is determined by $(a_0, a_1)$ because of the recursion relation. On the other hand, for every $(a_0, a_1)$ there is a series with these initial terms, because we just define $a_n$ recursively. (This is not a proof, just a heuristic.) □

(2) Which of the following vectors when added to the set $\{(1,2,3,1),(2,2,1,3)\}$ makes it linearly dependent? (the coordinates are real numbers)
(a) $(3,4,1,6)$.
(b) $(-1,2,7,-3)$. $\Leftarrow\Leftarrow\Leftarrow$
(c) $(-1,-2,4,2)$.

*Proof.* One just tries for each vector in turn to find $x, y$ such that that vector is $x(1,2,3,1) + y(2,2,1,3)$. □

(3) Is there a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ whose kernel is $\{(a,b,c,d) : a+b=0\}$ and whose image is $\{(a,b,0,0) : a, b \in \mathbb{R}\}$?
(a) Yes.
(b) No. $\Leftarrow\Leftarrow\Leftarrow$

*Proof.* This would contradict $\dim \ker + \dim \operatorname{Im} = 4$, as the kernel is 3 dimensional and the image is 2 dimensional. □

(4) Consider the following two bases of $\mathbb{R}^2$, $B = \{(1,1),(1,0)\}$, $C = \{(0,1),(2,5)\}$. The change of basis matrix ${}_C M_B$ (the matrix such that for every vector $v$ one has $[v]_C = {}_C M_B [v]_B$) is:

(a) $\begin{pmatrix} 1 & 5 \\ -1 & -3 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 & 7 \\ 0 & 2 \end{pmatrix}$.

(c) $\begin{pmatrix} -3/2 & -5/2 \\ 1/2 & 1/2 \end{pmatrix}$. $\Leftarrow\Leftarrow\Leftarrow$

(d) $\begin{pmatrix} 2 & 0 \\ 6 & 1 \end{pmatrix}$.

*Proof.* We have ${}_{St}M_B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, ${}_{C}M_{St} = {}_{St}M_C^{-1} = \begin{pmatrix} 0 & 2 \\ 1 & 5 \end{pmatrix}^{-1} = \frac{-1}{2}\begin{pmatrix} 5 & -2 \\ -1 & 0 \end{pmatrix}$ and

$$ {}_{C}M_B = {}_{C}M_{St}\ {}_{St}M_B = \frac{-1}{2}\begin{pmatrix} 5 & -2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -3/2 & -5/2 \\ 1/2 & 1/2 \end{pmatrix}. $$

□

(5) Let $p$ be a prime number and $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ the field of $p$ elements. Let $T : \mathbb{F}^2 \to \mathbb{F}^2$ be the linear map

$$ T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. $$

For which $p$ is $T$ a nilpotent map?

(a) For no $p$.
(b) For $p = 3$ and 11.
(c) For $p = 2$ and 3.
(d) For $p = 2$.⇐⇐⇐

*Proof.* From what we've proved $T$ is nilpotent if and only if $T^2$ is the zero map (because the vector space is 2 dimensional). But $T^2$ is given by $\begin{pmatrix} 2 & 3 \\ 6 & -6 \end{pmatrix}^2 = \begin{pmatrix} 22 & -12 \\ -24 & -54 \end{pmatrix}$. This matrix is zero if and only if $p = 2$. □

PART II. (70% of grade)

(1) (a) State Steinitz's lemma.
(b) Using Steinitz's lemma, prove that if $V$ is a finite dimensional vector space then any two bases of $V$ have the same cardinality.

(2) Let $n \geq 2$ be an integer. Determine the dimension of the vector space

$$ V := \{f : \{1, 2, \dots, n\} \to \mathbb{R} : \sum_{i=1}^{n} f(i) = 0, \sum_{i=1}^{n} (-1)^i f(i) = 0\}. $$

Provide a full proof.

*Proof.* The dimension is $n - 2$. Let $W := \{f : \{1, 2, \dots, n\} \to \mathbb{R}\}$. From what we did in the assignments we know that $W$ has dimension $n$. Consider the map

$$ T : W \to \mathbb{R}^2, \quad T(f) = \left(\sum_{i=1}^{n} f(i),\ \sum_{i=1}^{n} (-1)^i f(i)\right). $$

We have $\ker(T) = V$. From the theorem about the kernel and the image, it is enough to prove that $\dim \mathrm{Im}(T) = 2$. For that, it is enough to show that there are 2 non-proportional vectors in $\mathrm{Im}(T)$. Consider the function $f$ which is 1 on 1 and zero otherwise. $T(f) = (1, -1)$. Consider the function $g$ which is 1 on 1 and on 2 and otherwise zero. Then $T(g) = (2, 0)$. □