Algebra II MATH 251 **Instructor: Dr. E. Goren.**

Assignment 8

To be submitted by March 21, 12:00

1. (A). Develop a formula for the best linear approximation for a series of points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),$$

where $x_i, y_i \in \mathbb{R}$ and $x_1 < x_2 < \cdots < x_n$ without assuming that $\sum x_i = 0$.

(B) What line do you get for the data

$$(0,2), (1,3), (2,3)?$$

2. Given a monic polynomial of degree n, $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$, show that there exists an $n \times n$ matrix A, such that

$$\Delta_A(t) = f(t).$$

(Suggestion: Look at past assignments.)

3. Find an explicit formula for a_n in the series $a_0, a_1, a_2, \ldots$ given by

$$0,\ 1,\ 1,\ 3,\ 5,\ 11,\ 21,\ 43,\ \ldots.$$

4. **Markov Processes.** Imagine a particle that can be in any one of n states

$$S_1, S_2, \ldots, S_n.$$

(These may be the spin states of a particle, or the room my kid is in...) Initially, we might not know exactly the state the particle is in and have at our disposal only the probability that it is in a certain state, that is, we have a *probability distribution*

$$(f_1, f_2, \ldots, f_n), \quad f_i \geq 0, \quad f_1 + f_2 + \cdots + f_n = 1.$$

A certain process is now taking place (it is an example of certain stochastic processes called *Markov chains*). At any time $t = 1, 2, 3, \ldots$ the particle may change its state. We know the probability it changes its state from state S_j to state S_i and construct a matrix $M = (m_{ij})$, where m_{ij} is the probability that the particle goes from state S_j to state S_i.

(1) Show that M is a matrix of non-negative real numbers such that for every j we have $\sum_{i=1}^n m_{ij} = 1$.

(2) Prove that the probability distribution describing the state of the particle at time n is given by

$$M^n \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

(3) Consider the case of two states S_1, S_2 where M (called the *transition matrix*) is given by

$$\begin{pmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{pmatrix}.$$

Show that for any initial probability distribution (f_1, f_2) the limit

$$\lim_{n \to \infty} M^n \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

exists, and compute it.

5. Find the characteristic polynomial, eigenvalues λ and a basis for E_λ for the following matrices:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix},$$

over the fields: (i) $\mathbb{R}$, (ii) $\mathbb{C}$, (iii) $\mathbb{Z}/2\mathbb{Z}$, (iv) $\mathbb{Z}/5\mathbb{Z}$.

6. Calculate the characteristic and minimal polynomial of the following matrices with real entries. In each case determine the algebraic and geometric multiplicity of each eigenvalue. Decide which matrix is diagonalizable and for that one, say A, find an invertible matrix M such that $M^{-1}AM$ is diagonal.

$$\begin{pmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{pmatrix}$$

7. Let A be a matrix in block form:

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \\ & & \ddots & \\ 0 & 0 & \cdots & A_k \end{pmatrix}.$$

Prove that

$$\Delta_A = \Delta_{A_1} \Delta_{A_2} \cdots \Delta_{A_r},$$

and

$$m_A = \mathrm{lcm}\{m_{A_1}, m_{A_2}, \cdots, m_{A_r}\}.$$

You may use the formula

$$A^b = \begin{pmatrix} A_1^b & 0 & \cdots & 0 \\ 0 & A_2^b & & \\ & & \ddots & \\ 0 & 0 & \cdots & A_k^b \end{pmatrix}$$

for every positive integer b.