Algebra II, MATH 251 **Instructor: Dr. E. Goren.**

Assignment 5

To be submitted by February 14, 12:00

1. Deduce from the theorems on determinants the following:
   (1) If a column is zero, the determinant is zero.
   (2) $\det(A) = \det(A^t)$, where A^t is the transposed matrix.
   (3) If a row is zero, the determinant is zero.
   (4) Let A be a matrix in "upper diagonal block form":
   $$A = \begin{pmatrix} A_1 & & & \star \\ 0 & A_2 & & \\ & & \ddots & \\ 0 & & 0 & A_k \end{pmatrix}.$$
   Here each A_i is a square matrix say of size r_i, and A_2 starts at the $r_1 + 1$ column and $r_1 + 1$ row, etc. Prove that
   $$\det(A) = \det(A_1)\det(A_2)\cdots\det(A_k).$$
   Conclude that the determinant of an upper triangular matrix is given by
   $$\det \begin{pmatrix} a_{11} & & & \star \\ 0 & a_{22} & & \\ & & \ddots & \\ 0 & & 0 & a_{kk} \end{pmatrix} = a_{11}a_{22}\cdots a_{kk}.$$
   (Here each a_{ii} is a scalar).
2. Calculate the following series of determinants.
   (1) $\det\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\det\begin{pmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{pmatrix}$, $\det\begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix}, \cdots$
   (2) $\det\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, $\det\begin{pmatrix} 1&1&0 \\ 1&1&1 \\ 0&1&1 \end{pmatrix}$, $\det\begin{pmatrix} 1&1&0&0 \\ 1&1&1&0 \\ 0&1&1&1 \\ 0&0&1&1 \end{pmatrix}$, $\det\begin{pmatrix} 1&1&0&0&0 \\ 1&1&1&0&0 \\ 0&1&1&1&0 \\ 0&0&1&1&1 \\ 0&0&0&1&1 \end{pmatrix}, \cdots$
   (3) $\det\begin{pmatrix} x & -a_2 \\ 1 & x+a_1 \end{pmatrix}$, $\det\begin{pmatrix} x&0&a_3 \\ 1&x&-a_2 \\ 0&1&x+a_1 \end{pmatrix}$, $\det\begin{pmatrix} x&0&0&-a_4 \\ 1&x&0&a_3 \\ 0&1&x&-a_2 \\ 0&0&1&x+a_1 \end{pmatrix}, \cdots$
3. Prove the following formula (the *Vandermonde determinant*):
   $$\det\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots x_1^{n-1} \\ \vdots & & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots x_n^{n-1} \end{pmatrix} = \prod_{i>j}(x_i - x_j)$$

For example, for $n = 2, 3$ we have
$$\det\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} = (x_2 - x_1), \quad \det\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

I encourage experimenting with the computer to answer question 2. Here is how you calculate determinants in Maple:

Input:	with(linalg):
Input:	A := matrix([[1,1,0],[1,1,1], [0,1,1]]);
Output:	A := matrix([[1, 1, 0], [1, 1, 1], [0, 1, 1]])
Input:	det(A);
Output:	-1
Input:	A:= matrix([[x,y,0],[1,z,1], [0,1,1]]);
Output:	A := matrix([[x, y, 0], [1, z, 1], [0, 1, 1]])
Input:	det(A);
Output:	x*z-x-y
Input:	B:= inverse(A);
Output:	matrix([[(z-1)/(x*z-x-y), -y/(x*z-x-y), y/(x*z-x-y)], [-1/(x*z-x-y), x/(x*z-x-y), -x/(x*z-x-y)], [1/(x*z-x-y), -x/(x*z-x-y), (x*z-y)/(x*z-x-y)]])
Input:	det(B)
Output:	1/(x*z-x-y)

Bonus question - 15 points. *Reed-Solomon Codes.* Let $\mathbb{F}$ be a finite field with q elements. List the non-zero elements of $\mathbb{F}_q$ as $\{\beta_1, \dots, \beta_{q-1}\}$. Define a map

$$\mathbb{F}_q[x]_k \longrightarrow \mathbb{F}^{q-1},$$

by

$$f \mapsto T(f) := (f(\beta_1), \dots, f(\beta_{q-1})).$$

Prove that T is a linear map and find when is it injective. When this holds, the image of T is a (n, k) code. Find the minimal Hamming weight of a non-zero element of the code. Compare your result with Assignment 2.