Assignment 2

1. Let $V$ be the set of all sequences of complex numbers $(a_0, a_1, a_2, \ldots)$ satisfying
   
   \[ a_n = a_{n-1} + a_{n-2}, \quad \forall n \geq 2. \]

   Show that $V$ has a natural structure of a vector space over $\mathbb{C}$. Find its dimension and a basis.

2. Let $V$ be an $n$-dimensional vector space over a field $\mathbb{F}$. Let $T = \{t_1, \ldots, t_m\} \subset V$ be a linearly independent set. Let $W = \text{Span}(T)$. Prove:
   
   \[ \dim(W) = m. \]

3. Let $W$ be a subspace of a vector space $V$ of dimension $n$. Let $\{t_1, \ldots, t_m\}$ be a basis for $W$. Prove that there exist vectors $\{t_{m+1}, \ldots, t_n\}$ in $V$ such that $\{t_1, \ldots, t_m, t_{m+1}, \ldots, t_n\}$ is a basis for $V$.

4. Let $V_1, V_2$ be finite dimensional vector spaces over a field $\mathbb{F}$. Prove that
   
   \[ \dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2). \]

5. Consider $V := \mathbb{R}[t]_n$, the vector space of polynomials of degree $< n$ with real coefficients. Let
   
   \[ r_1 < r_2 < \cdots < r_n \]

   be any real numbers. Show that for every $i$ there exists a unique polynomial $f_i$ in $V$ that vanishes at all the $r_j$ except for $r_i$ where it obtains the value 1. Give an explicit formula for $f_i$. Show that
   
   \[ f_1, f_2, \ldots, f_n \]

   comprise a basis for $V$.

6. Let $\mathcal{B} = \{(1, 1), (1, 5)\}$ and $\mathcal{C} = \{(2, 1), (1, -1)\}$ be bases of $\mathbb{R}^2$. Find the change of basis matrices $g_M_c$ and $c_M_B$ between the bases $\mathcal{B}$ and $\mathcal{C}$. Let $v = \begin{pmatrix} 5 \\ 23 \end{pmatrix}$ with respect to the standard basis. Find $[v]_\mathcal{B}$ and $[v]_\mathcal{C}$.

7. Let $\mathbb{F}$ be a finite field with $q$ elements.
   
   (1) Show that the kernel of the ring homomorphism

   \[ \mathbb{Z} \rightarrow \mathbb{F} \]

   defined by $n \mapsto n \cdot 1 = 1 + \cdots + 1$ ($n$ times) is of the form $p\mathbb{Z}$ for some prime $p$. Conclude that we may assume that $\mathbb{F} \cong \mathbb{Z}/p\mathbb{Z}$ for some prime $p$.

   (2) Prove that $\mathbb{F}$ is a vector space of finite dimension over $\mathbb{Z}/p\mathbb{Z}$ and if this dimension is $n$ then $\mathbb{F}$ has $p^n$ elements\(^1\).

**Bonus question** ( = 20%). Let $\mathbb{F}$ be a finite field of $q$ elements. Let $V = \mathbb{F}^n$ and let $C$ be a code (= a subspace) of dimension $k$, hence having $q^k$ elements. Let $d$ be the minimal Hamming weight of a non zero element of $C$. Prove that

\[ d \leq n - k + 1. \]

\(^1\)Note: at this point you’ve proven that every finite field has cardinality $p^n$ for some prime $p$. 
