1. Let \( v_1, \ldots, v_n \) be a basis for \( V \) and let \( T : V \to V \) be a linear transformation. For every \( 1 \leq i \leq n \) we define a polynomial \( P_i \) as follows: Let \( r \) be the first integer so that \( v_i, T v_i, \ldots, T^r v_i \) are linearly dependent. Then for suitable scalars \( b_0, \ldots, b_r \) we have
\[
 b_r T^r v_i + \cdots + b_1 T v_i + b_0 = 0.
\]
We may assume without loss of generality that \( b_r = 1 \). Let
\[
P_i(t) = t^r + b_{r-1} t^{r-1} + \cdots + b_1 t + b_0 \quad \text{(the } b_j \text{ depend on } i \text{ but we do not make this dependence explicit in the notation).}
\]
Let \( P = \text{lcm}\{P_1, \ldots, P_n\} \). Prove that \( P \) is the minimal polynomial of \( T \).

2. Prove that the matrix of complex numbers
\[
 A = \begin{pmatrix} 1 & 1 & 5 & 0 & 0 \\ 7 & 1 & 3 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}
\]
is diagonalizable. Note: you don’t need to diagonalize it to prove it !!

3. Determine the possibilities for the Jordan canonical form of a matrix \( A \) with characteristic polynomial \( \Delta_A(t) = (t - 1)^6(t - 2)^4(t - 4)^5 \) and minimal polynomial \( m_A(t) = (t - 1)^3(t - 2)^2(t - 4) \).

4. Find the Jordan canonical form of the matrix
\[
 A = \begin{pmatrix} 5 & 9 & -2 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}.
\]

5. It is known that a differentiable function \( f : \mathbb{R}^2 \to \mathbb{R} \) has a
- **maximum** at a point \( P \) if \( \partial f / \partial x = \partial f / \partial y = 0 \) at \( P \) and the \( 2 \times 2 \) symmetric matrix
\[
 \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}
\]
is positive definite;
- **minimum** at a point \( P \) if \( \partial f / \partial x = \partial f / \partial y = 0 \) at \( P \) and the \( 2 \times 2 \) symmetric matrix
\[
 \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}
\]
is positive definite;
• **saddle point** at $P$ if the $2 \times 2$ symmetric matrix

$$
\begin{pmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2}
\end{pmatrix}
$$

has one negative eigenvalue and one positive eigenvalue.

If $P$ is either a maximum, minimum or saddle point, we call it a simple critical point. Determine the nature of the simple critical point of the following functions at the origin $(0,0)$

$$
f(x, y) = 2x^2 + 6xy + y^2, \quad f(x, y) = x \sin(x) - \cos(y) - xy.
$$

(You may view the graphs and rotate them in Maple using

plot3d($2x^2 + 6xy + y^2$, x= -4..4, y = -4..4);
plot3d($x \sin(x) - \cos(y) -xy$, x= -4..4, y = -4..4, numpoints=3000);)

**Remark.** This criterion can be generalized to functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$. If all the first partials vanish at a point $P$ and the matrix of mixed derivatives is positive definite (resp. negative definite) at $P$, then the function has a minimum (resp. maximum) at $P$.

6. Find a formula for the general term of the sequence

$$
0, 1, 4, 12, 32, 80, \ldots
$$

$$
(a_{n+2} = 4a_{n+1} - 4a_n).
$$

**Note:** The point here would be that the matrix $A$ that comes out is not diagonalizable. Still you should find some matrix $D = MAM^{-1}$ simple enough so that the trick $A^N = M^{-1}D^NM$ is still useful.