The sign of a permutation, and realizing permutations as linear transformations.

Lemma 1. *Let $n \geq 2$. Let S_n be the group of permutations of $\{1, 2, \ldots, n\}$. There exists a surjective homomorphism of groups*

$$\mathbf{sgn} : S_n \longrightarrow \{\pm 1\}$$

(called the 'sign'). It has the property that for every $i \neq j$,

$$\mathbf{sgn}(\ (ij)\ ) = -1.$$

Proof. Consider the polynomial in n-variables[1]

$$p(x_1, \ldots, x_n) = \prod_{i<j}(x_i - x_j).$$

Given a permutation σ we may define a new polynomial

$$\prod_{i<j}(x_{\sigma(i)} - x_{\sigma(j)}).$$

Note that $\sigma(i) \neq \sigma(j)$ and for any pair $k < \ell$ we obtain in the new product either $(x_k - x_\ell)$ or $(x_\ell - x_k)$. Thus, for a suitable choice of sign $\mathbf{sgn}(\sigma) \in \{\pm 1\}$, we have[2]

$$\prod_{i<j}(x_{\sigma(i)} - x_{\sigma(j)}) = \mathbf{sgn}(\sigma) \prod_{i<j}(x_i - x_j).$$

We obtain a function

$$\mathbf{sgn} : S_n \longrightarrow \{\pm 1\}.$$

This function satisfies $\mathbf{sgn}(\ (k\ell)\ ) = -1$ (for $k < \ell$): Let $\sigma = (k\ell)$ and consider the product

$$\prod_{i<j}(x_{\sigma(i)} - x_{\sigma(j)}) = (x_\ell - x_k) \prod_{\substack{i<j \\ i\neq k, j\neq \ell}} (x_i - x_j) \prod_{\substack{k<j \\ j\neq \ell}} (x_\ell - x_j) \prod_{\substack{i<\ell \\ i\neq k}} (x_i - x_k)$$

Counting the number of signs that change we find that

$$\prod_{i<j}(x_{\sigma(i)} - x_{\sigma(j)}) = (-1)(-1)^{\sharp\{j:k<j<\ell\}}(-1)^{\sharp\{i:k<i<\ell\}} \prod_{i<j}(x_i - x_j) = -\prod_{i<j}(x_i - x_j).$$

It remains to show that $\mathbf{sgn}$ is a group homomorphism. We first make the innocuous observation that for *any* variables $y_1, \ldots, y_n$ and for *any* permuation σ we have

$$\prod_{i<j}(y_{\sigma(i)} - y_{\sigma(j)}) = \mathbf{sgn}(\sigma) \prod_{i<j}(y_i - y_j).$$

Let τ be a permutation. We apply this observation for the variables $y_i := x_{\tau(i)}$. We get

$$\begin{aligned} \mathbf{sgn}(\tau\sigma)p(x_1, \ldots, x_n) &= p(x_{\tau\sigma(1)}, \ldots, x_{\tau\sigma(n)}) \\ &= p(y_{\sigma(1)}, \ldots, y_{\sigma(n)}) \\ &= \mathbf{sgn}(\sigma)p(y_1, \ldots, y_n) \\ &= \mathbf{sgn}(\sigma)p(x_{\tau(1)}, \ldots, x_{\tau(n)}) \\ &= \mathbf{sgn}(\sigma)\mathbf{sgn}(\tau)p(x_1, \ldots, x_n). \end{aligned}$$

---

[1] For $n = 2$ we get $x_1 - x_2$. For $n = 3$ we get $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$.

[2] For example, if $n = 3$ and σ is the cycle (123) we have

$$(x_{\sigma(1)} - x_{\sigma(2)})(x_{\sigma(1)} - x_{\sigma(3)})(x_{\sigma(2)} - x_{\sigma(3)}) = (x_2 - x_3)(x_2 - x_1)(x_3 - x_1) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

Hence, $\mathbf{sgn}(\ (1\ 2\ 3)\ ) = 1$.

This gives

$$\mathbf{sgn}(\tau\sigma) = \mathbf{sgn}(\tau)\mathbf{sgn}(\sigma).$$

$\square$

Calculating sgn in practice. Recall that every permutation σ can be written as a product of disjoint cycles

$$\sigma = (a_1 \dots a_\ell)(b_1 \dots b_m) \dots (f_1 \dots f_n).$$

Claim: $\mathbf{sgn}(a_1 \dots a_\ell) = (-1)^{\ell-1}$.
Corollary: $\mathbf{sgn}(\sigma) = (-1)^{\sharp \text{ even length cycles}}$.

Proof. We write

$$(a_1 \dots a_\ell) = \underbrace{(a_1 a_\ell) \dots (a_1 a_3)(a_1 a_2)}_{\ell-1 \text{ transpositions}}.$$

Since a transposition has sign -1 and **sgn** is a homomorphism, the claim follows. $\square$

A Numerical example. Let $n = 11$ and

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 5 & 4 & 3 & 1 & 7 & 8 & 10 & 6 & 9 \end{pmatrix}.$$

Then

$$\sigma = (1\ 2\ 5)(3\ 4)(6\ 7\ 8\ 10\ 9).$$

Now,

$$\mathbf{sgn}(\ (1\ 2\ 5)\ ) = 1, \quad \mathbf{sgn}(\ (3\ 4)\ ) = -1, \quad \mathbf{sgn}(\ (6\ 7\ 8\ 10\ 9)\ ) = 1.$$

We conclude that $\mathbf{sgn}(\sigma) = -1$.

Realizing S_n as linear transformations. Let $\mathbb{F}$ be any field. Let $\sigma \in S_n$. There is a unique linear transformation

$$T_\sigma : \mathbb{F}^n \longrightarrow \mathbb{F}^n,$$

such that

$$T(e_i) = e_{\sigma(i)}, \quad i = 1, \dots n,$$

where, as usual, $e_1, \dots, e_n$ are the standard basis of $\mathbb{F}^n$. Note that

$$T_\sigma \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma^{-1}(1)} \\ x_{\sigma^{-1}(2)} \\ \vdots \\ x_{\sigma^{-1}(n)} \end{pmatrix}.$$

(For example, because $T_\sigma x_1 e_1 = x_1 e_{\sigma(1)}$, the $\sigma(1)$ coordinate is x_1, namely, in the $\sigma(1)$ place we have the entry $x_{\sigma^{-1}(\sigma(1))}$.) Since for every i we have $T_\sigma T_\tau(e_i) = T_\sigma e_{\tau(i)} = e_{\sigma\tau(i)} = T_{\sigma\tau} e_i$, we have the relation

$$T_\sigma T_\tau = T_{\sigma\tau}.$$

The matrix representing T_σ is the matrix (a_{ij}) with $a_{ij} = 0$ unless $i = \sigma(j)$. For example, for $n = 4$ the matrices representing the permutations $(12)(34)$ and $(1\ 2\ 3\ 4)$ are, respectively

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Otherwise said,[3]

$$T_\sigma = \left(e_{\sigma(1)} \;\middle|\; e_{\sigma(2)} \;\middle|\; \dots \;\middle|\; e_{\sigma(n)}\right) = \begin{pmatrix} e_{\sigma^{-1}(1)} \\ \hline e_{\sigma^{-1}(2)} \\ \hline \vdots \\ \hline e_{\sigma^{-1}(n)} \end{pmatrix}.$$

It follows that

$$\begin{aligned} \mathbf{sgn}(\sigma)\det(T_\sigma) &= \mathbf{sgn}(\sigma)\det\left(e_{\sigma(1)} \;\middle|\; e_{\sigma(2)} \;\middle|\; \dots \;\middle|\; e_{\sigma(n)}\right) \\ &= \det\left(e_1 \;\middle|\; e_2 \;\middle|\; \dots \;\middle|\; e_n\right) \\ &= \det(I_n) \\ &= 1. \end{aligned}$$

Recall that $\mathbf{sgn}(\sigma) \in \{\pm 1\}$. We get

$$\det(T_\sigma) = \mathbf{sgn}(\sigma).$$

[3]This gives the interesting relation $T_{\sigma^{-1}} = T_\sigma^t$. Because $\sigma \mapsto T_\sigma$ is a group homomorphism we may conclude that $T_\sigma^{-1} = T_\sigma^t$. Of course for a general matrix this doesn't hold.