Algebra II 189-251B **Instructor: Dr. E. Goren.**

Midterm Exam. February 26, 2001.

Calculators, books or notes are not allowed. Answer all questions in Part A and B. Part C is optional.

Notation. The symbols $\mathbb{Z}, \mathbb{R}$, stand for the integers and real numbers, respectively. If $\mathbb{F}$ is a field we denote by $\mathbb{F}[t]_n$ the vector space over $\mathbb{F}$ of polynomials with coefficients in $\mathbb{F}$ and of degree *less* than n.

PART A. Each question has a unique correct answer. Give only that answer number with no explanations or proofs. Each question is worth 8 points.

1. There exists a surjective linear transformation from $\mathbb{R}^5$ to $\mathbb{R}[t]_6$.
   1. Yes.
   2. No.

2. Let $\mathbb{F}$ be the field $\mathbb{Z}/7\mathbb{Z}$. There exists a non-zero polynomial $f \in \mathbb{F}[t]_4$ such that

$$f(0) = 0, \quad f(1) + f(3) = 0, \quad f'(0) = 0.$$

   1. Yes.
   2. No.

3. Let A be an invertible $n \times n$ matrix with entries in a field $\mathbb{F}$. Let St be the standard basis for $\mathbb{F}^n$. There exists a basis $\mathcal{B}$ such that $A = {}_{St}M_{\mathcal{B}}$.
   1. Yes.
   2. No.

4. The rank of the matrix $\begin{pmatrix} 1 & 3 & 6 \\ 1 & 2 & 3 \\ 1 & 5 & 0 \end{pmatrix}$ over the field $\mathbb{Z}/3\mathbb{Z}$ is
   1. 0.
   2. 1.
   3. 2.
   4. 3.

5. Let $\mathcal{B} = \{(12, 13), (-11, 3)\}$ be a basis of $\mathbb{R}^2$. There exists a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that

$${}_{St}[T]_{St} = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}, \quad {}_{\mathcal{B}}[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

   1. Yes.
   2. No.

PART B. Answer all questions. Each question is worth 30 points.

1. (A) Prove that a maximal independent set is a spanning set (no need to prove it is minimal).
   (B) Prove that for any two $n \times n$ matrices A, B we have
$$\det(AB) = \det(A)\det(B).$$

2. Let W be the subspace of $\mathbb{R}^4$ consisting of the solutions to the equation
$$x_1 + x_2 + x_3 - x_4 = 0.$$
Let U be the subspace of $\mathbb{R}^4$ equal to the span of the vector $(1, 1, 1, -1)$.

Find the projection on W along U (write it as a matrix with respect to the standard basis). What is the image of the vector $(1, -1, 1, -1)$ under this projection?

PART C. This part is optional and is worth 15 points.

Let V be a vector space over a field $\mathbb{F}$, perhaps of infinite dimension. We say that a linear operator $T : V \longrightarrow V$ is nilpotent if for every $v \in V$ there exists an integer $n(v) \geq 1$ such that $T^{n(v)}(v) = 0$. Show

1. If V has finite dimension this is equivalent to the definition of a nilpotent operator that we gave, namely, there exists an integer $N \geq 1$ such that $T^N \equiv 0$.
2. Give an example showing Fitting's lemma fails for infinite dimensional vector spaces.