**Algebra II 189-251B** **Instructor: Dr. E. Goren.**

# Assignment 9

**To be submitted by March 30, 12:00**

1. Let $S$ be a linear operator and $\mu_1, \dots, \mu_r$ be distinct eigenvalues of $S$. Let $E_{\mu_1}, \dots, E_{\mu_r}$ be the corresponding eigenspaces. Let $v_i \in E_{\mu_i}$ be a non-zero vector. Prove that the set $\{v_1, \dots, v_r\}$ is linearly independent.

<u>Hint</u>: Assume linear dependence. Let $v_k$ be the *first* vector that is linearly dependent on the preceding vectors. Write this equation explicitly and apply $S$. Try to get a linear dependence of some $v_r$ (for $r < k$) on the preceding vectors and conclude a contradiction.

2. Calculate the characteristic and minimal polynomial of the following matrices with real entries. In each case determine the algebraic and geometric multiplicity of each eigenvalue. Decide which matrix is diagonalizable and for that one, say $A$, find an invertible matrix $M$ such that $M^{-1}AM$ is diagonal.

$$\begin{pmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{pmatrix}$$

3. For each matrix $N$ in Exercise 2 (considered as a linear transformation $T$) find the Primary Decomposition, i.e., the factorization of the minimal polynomial, the kernels of the factors, and for each kernel a matrix representation of $T$.

4. Let $V$ be a vector space and let $W_1, \dots, W_r$ be subspaces of $V$. Prove that the following are equivalent:

(1) Every vector $v$ can be written uniquely as $v = w_1 + \cdots + w_r$ with $w_i \in W_i$.
(2) If $\mathcal{B}_i$ is a basis for $W_i$ then $\cup_{i=1}^r \mathcal{B}_i$ is a basis for $V$.
(3) Every vector $v$ can be written as $v = w_1 + \cdots + w_r$ with $w_i \in W_i$ and $\dim(V) = \dim(W_1) + \cdots + \dim(W_r)$.

5. Let $A$ be a matrix in block form:

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \\ & & \ddots & \\ 0 & 0 & \cdots & A_k \end{pmatrix}.$$

Prove that

$$\Delta_A = \Delta_{A_1} \Delta_{A_2} \cdots \Delta_{A_r},$$

and

$$m_A = \mathrm{lcm}\{m_{A_1}, m_{A_2}, \cdots, m_{A_r}\}.$$

You may use the formula

$$A^b = \begin{pmatrix} A_1^b & 0 & \cdots & 0 \\ 0 & A_2^b & & \\ & & \ddots & \\ 0 & 0 & \cdots & A_k^b \end{pmatrix}$$

for every positive integer $b$.

6. Let $S$ and $T$ be commuting linear maps from a vector space $V$ to itself. Let $\lambda$ be an eigenvalue of $T$ and let $E_\lambda$ be the corresponding eigenspace. Prove that $E_\lambda$ is $S$ invariant. Conclude that if $T$ is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_r$, and therefore

$$V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_r},$$

we may decompose $S$ as

$$S = S_1 \oplus \cdots \oplus S_r,$$

where $S_i : E_{\lambda_i} \longrightarrow E_{\lambda_i}$.

**Remark**: This problem will be continued in the next assignment. The final statement will be that two commuting diagonalizable linear operators may be diagonalized *simultaneously*. This is used very often in applications.