Algebra II 189-251B **Instructor: Dr. E. Goren.**

Assignment 8

To be submitted by March 23, 12:00

1. Let W be the subspace of $\mathbb{F}^4$ defined by the equation $x_1 + x_2 + x_3 + x_4 = 0$. Find the orthogonal projection of $(1, 0, 0, 0)$ on W. Remark: I would allow here solutions that use Maple (or other software) if you include a print out of the commands you used.

2. (A). Develop a formula for the best linear approximation for a series of points

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n),$$

where $x_i, y_i \in \mathbb{R}$ and $x_1 < x_2 < \cdots < x_n$ without assuming that $\sum x_i = 0$.

(B) What line do you get for the data

$$(0, 2), (1, 3), (2, 3)?$$

3. Given a monic polynomial of degree n, $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$, show that there exists an $n \times n$ matrix A, such that

$$\Delta_A(t) = f(t).$$

(Suggestion: Look at past assignments.)

4. Find an explicit formula for a_n in the series $a_0, a_1, a_2, \dots$ given by

$$0,\ 1,\ 1,\ 3,\ 5,\ 11,\ 21,\ 43,\ \dots.$$

5. **Markov Processes.** Imagine a particle that can be in any one of n states

$$S_1, S_2, \dots, S_n.$$

(These may be the spin states of a particle, or the room my kid is in...) Initially, we might not know exactly the state the particle is in and have at our disposal only the probability it is in a certain state, that is, we have a *probability distribution*

$$(f_1, f_2, \dots, f_n), \quad f_i \geq 0, \quad f_1 + f_2 + \cdots + f_n = 1.$$

A certain process is now taking place (it is an example of certain stochastic processes called *Markov chains*). At every time $t = 1, 2, 3, \dots$ the particle may change its state. We know the probability it change its state from state S_j to state S_i and construct a matrix $M = (m_{ij})$, where m_{ij} is the probability that the particle goes from state S_j to state S_i.

(1) Show that M is a matrix of non-negative real numbers such that for every j we have $\sum_{i=1}^n m_{ij} = 1$.

(2) Prove that the probability distribution describing the state of the particle at time n is given by

$$M^n \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

(3) Consider the case of two states S_1, S_2 where M (called the *transition matrix*) is given by

$$\begin{pmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{pmatrix}.$$

Show that for any initial probability distribution (f_1, f_2) the limit

$$\lim_{n \to \infty} M^n \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

exists, and compute it.

6. Find the characteristic polynomial, eigenvalues λ and a basis for E_λ for the following matrices:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix},$$

over the fields: (i) $\mathbb{R}$, (ii) $\mathbb{C}$, (iii) $\mathbb{Z}/2\mathbb{Z}$, (iv) $\mathbb{Z}/5\mathbb{Z}$.