Algebra II 189-251B **Instructor: Dr. E. Goren.**

Assignment 7

To be submitted by March 16, 12:00
In this assignment V is an inner product space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

1. Prove the Parallelogram Law:
$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

2. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix of complex numbers. Prove that the function
$$\langle (x_1, x_2), (y_1, y_2) \rangle = (x_1, x_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \end{pmatrix},$$
is an inner product on $\mathbb{C}^2$ if and only if a and d are positive real numbers, $c = \bar{b}$ and $ad - b\bar{b} > 0$.
In the case of the matrix $\left(\begin{smallmatrix} 1 & 1+i \\ 1-i & 5 \end{smallmatrix}\right)$ compute $\langle (1,2), (3,4) \rangle$ and $\|(2, 5i)\|$.

3. Let $a < b$ be real numbers. Show that the function
$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$
defines an inner product of $\mathbb{R}[x]_n$.[1] Compute the norm of the vector $f(x) = 1 + x + x^2$ in the case $(a, b) = (0, 1)$ and in the case $(a, b) = (0, 2)$.

4. Find an orthonormal basis for $\mathbb{C}^2$ with the inner product defined by the matrix $\left(\begin{smallmatrix} 1 & 1+i \\ 1-i & 5 \end{smallmatrix}\right)$.

5. Perform the Gram-Schmidt process for the basis $\{1, x, x^2\}$ to $\mathbb{R}[x]_3$ with respect to the inner product
$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx.$$

6. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{F}^n$. Define the unit ball in $\mathbb{F}^n$ to be the set
$$\mathbb{B} = \{(x_1, \dots, x_n) : \|(x_1, \dots, x_n)\| \leq 1\}.$$
Show that this is a *central convex body*:
(1) $0 \in \mathbb{B}$.
(2) if $\alpha \in [0, 1]$ and $u, v \in \mathbb{B}$ then $\alpha u + (1 - \alpha)v \in \mathbb{B}$;
(3) if $|\alpha| \leq 1$ then $v \in \mathbb{B} \Rightarrow \alpha v \in \mathbb{B}$.

A difficult question to the valiant among us: Which convex bodies actually arise from inner products?!

[1] This is also true for $\mathbb{C}[x]_n$ if we define
$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx.$$
Note that in this case one can do the integration formally because we are dealing with polynomials. Thus, for example,
$$\langle x^2, 1 + ix \rangle = \int_a^b (x^2 - ix^3)\, dx = (x^3/3 - ix^4/4)|_a^b = (b^3/3 - ib^4/4) - (a^3/3 - ia^4/4).$$