Algebra II 189-251B **Instructor: Dr. E. Goren.**

Assignment 6

To be submitted by March 9, 12:00

1. Let V be a finite dimensional vector space and let $W \subseteq V$ be a subspace. Prove that

$$\dim(V) = \dim(W) + \dim(V/W).$$

2. Define functionals $\phi_1, \dots, \phi_n$ on $\mathbb{R}[t]_n$ by

$$\phi_i(f) = f(i).$$

Prove that $\{\phi_1, \dots, \phi_n\}$ is a basis for $(\mathbb{R}[t]_n)^\star$. What basis $\{s_1, \dots, s_n\}$ for $\mathbb{R}[t]_n$ has the property that $\{\phi_1, \dots, \phi_n\}$ is its dual basis?

3. Let U_1, U_2 be subspaces of a vector space V. Prove that[1]

$$(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp, \quad (U_1 \cap U_2)^\perp = U_1^\perp + U_2^\perp,$$

4. Let $T : V \longrightarrow W$ be a linear transformation with kernel U. Show that $\mathrm{Im}(T^\star) = U^\perp$. Use that to conclude the following facts:[2]

(1) If T is injective, $T^\star$ is surjective.
(2) If T is surjective, $T^\star$ is injective. (I suggest proving this fact directly from the definitions and *not* as a consequence of $\mathrm{Im}(T^\star) = U^\perp$).
(3) $\dim(\mathrm{Im}(T)) = \dim(\mathrm{Im}(T^\star))$.
(4) For every $m \times n$ matrix A we have

$$\mathrm{rank}_r(A) = \mathrm{rank}_c(A).$$

[1]In your text book the notation U° is used instead of $U^\perp$.
[2]Try and prove the facts even if you can not prove that $\mathrm{Im}(T^\star) = U^\perp$.