1. Let

\[ W_1 = \text{Span}(\{(1, 1, 0, 0), (0, 1, 1, -1)\}) \]
\[ W_2 = \text{Span}(\{(3, 2, 1, 0), (4, 4, 2, -1)\}) \]

Find a system of homogeneous linear equations such that \(W_1\) is their solutions. The same for \(W_2\). Find then \(W_1 \cap W_2\).

2. (i) Compute the rank of the following system of linear equations, by finding the reduced echelon form.

\[
\begin{align*}
5x_1 + x_2 + 3x_3 + 2x_4 + x_5 &= b_1 \\
x_1 + x_3 + x_4 &= b_2 \\
7x_1 + 2x_2 + 3x_3 + x_4 + 2x_5 &= b_3 \\
-x_1 - x_2 + x_3 + 2x_4 - x_5 &= b_4
\end{align*}
\]

(ii) Prove that at least for one of the vectors

\[
\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\]

we can’t solve the system.

(iii) Prove that there exists a non-trivial solution for the homogeneous system satisfying also \(x_1 = x_4\).

3. (1) Prove that the determinant of an \(n \times n\) matrix \(A = (a_{ij})\) is non-zero if and only if the columns of \(A\),

\[
\begin{pmatrix}
 a_{11} \\
 a_{21} \\
 \vdots \\
 a_{n1}
\end{pmatrix}, \ldots, \begin{pmatrix}
 a_{1n} \\
 a_{2n} \\
 \vdots \\
 a_{nn}
\end{pmatrix},
\]

form a basis for \(\mathbb{F}^n\).

(2) Generalize this as follows. Let \(A = (a_{ij})\) be an \(m \times n\) matrix. Prove that \(\text{rank}(A) = k\) if and only if there exists a \(k \times k\) sub-determinant of \(A\) that is not zero, and every \((k + 1) \times (k + 1)\) sub-determinant is zero.
By a $k \times k$ sub-determinant we mean the following. We choose $k$ columns $j_1 < j_2 < \cdots < j_k$ among the $n$ columns (so $k$ is assumed $\leq n$), and we choose $k$ rows $i_1 < i_2 < \cdots < i_k$ among the $m$ rows (so $k$ is assumed $\leq m$ as well). We then look at the $k \times k$ matrix $(a_{i_j j_m})$. Its determinant is what we call a $k \times k$ sub-determinant.

Example: Consider the matrix we tortured at the lectures
\[
\begin{pmatrix}
3 & 2 & 0 \\
1 & 1 & 1 \\
2 & 1 & -1
\end{pmatrix}
\]
We know it has rank 2. We note that the only $3 \times 3$ determinant is 0. It has nine $2 \times 2$ sub-determinants. Some of them are
\[
\det \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \det \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, \det \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \det \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}.
\]
The determinants are $1, -1, -2, 3$. Note that over every field the first determinant is non-zero. This shows that the matrix has rank 2. Note also that over the field $\mathbb{Z}/2\mathbb{Z}$ the before last determinant is zero, and over $\mathbb{Z}/3\mathbb{Z}$ the last determinant is zero. This shows that if some $k \times k$ sub-determinant is zero this need not hold for other $k \times k$ sub-determinants.

4. Give two more proofs of the following theorem

**Theorem 1.** Let $A$ be an $m \times n$ matrix. Then
\[
\text{rank}_c(A) = \text{rank}_r(A).
\]

(1) By applying Exercise 3 and properties of determinants.

(2) By completing the following argument. Write
\[
A = \begin{pmatrix}
R_1 \\
R_2 \\
\vdots \\
R_m
\end{pmatrix}.
\]
If $r = \text{rank}_r(A)$ then there exist vectors $S_1, \ldots, S_r$ in $\mathbb{F}^n$ such that for suitable $k_{ij}$
\[
A = \begin{pmatrix}
R_1 \\
R_2 \\
\vdots \\
R_m
\end{pmatrix} = \begin{pmatrix}
k_{11} & \cdots & k_{1r} \\
\cdots & \cdots & \cdots \\
k_{m1} & \cdots & k_{mr}
\end{pmatrix} \begin{pmatrix}
S_1 \\
S_2 \\
\vdots \\
S_r
\end{pmatrix}.
\]
(Explain why!). Now interpret each matrix as a linear transformation and see what you can say about the column rank!