Algebra II 189-251B **Instructor: Dr. E. Goren.**

Assignment 5

To be submitted by March 2, 12:00

Note: It is recommended that you use Maple or some other software to check, or even find, the results. However, the solution must always be a complete proof, independent of computer calculations.

1. Let

$$W_1 = \text{Span}\left(\{(1,1,0,0),(0,1,1,-1)\}\right)$$
$$W_2 = \text{Span}\left(\{(3,2,1,0),(4,4,2,-1)\}\right)$$

Find a system of homogeneous linear equations such that W_1 is their solutions. The same for W_2. Find then $W_1 \cap W_2$.

2. (i) Compute the rank of the following system of linear equations, by finding the reduced echelon form.

$$5x_1 + x_2 + 3x_3 + 2x_4 + x_5 = b_1$$
$$x_1 + x_3 + x_4 = b_2$$
$$7x_1 + 2x_2 + 3x_3 + x_4 + 2x_5 = b_3$$
$$-x_1 - x_2 + x_3 + 2x_4 - x_5 = b_4$$

(ii) Prove that at least for one of the vectors

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

we can't solve the system.
(iii) Prove that there exists a non-trivial solution for the homogeneous system satisfying also $x_1 = x_4$.

3.

(1) Prove that the determinant of an $n \times n$ matrix $A = (a_{ij})$ is non-zero if and only if the columns of A,

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix},$$

form a basis for $\mathbb{F}^n$.

(2) Generalize this as follows. Let $A = (a_{ij})$ be an $m \times n$ matrix. Prove that rank(A) $= k$ if and only if there exists a $k \times k$ sub-determinant of A that is not zero, and every $(k+1) \times (k+1)$ sub-determinant is zero.

By a $k \times k$ sub-determinant we mean the following. We choose k columns $j_1 < j_2 < \cdots < j_k$ among the n columns (so k is assumed $\leq n$), and we choose k rows $i_1 < i_2 < \cdots < i_k$ among the m rows (so k is assumed $\leq m$ as well). We then look at the $k \times k$ matrix $(a_{i_\ell j_m})$. Its determinant is what we call a $k \times k$ sub-determinant.

Example: Consider the matrix we tortured at the lectures

$$\begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

We know it has rank 2. We note that the only 3×3 *determinant is* 0. *It has nine* 2×2 *sub-determinants. Some of them are*

$$\det\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \det\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, \det\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \det\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}.$$

The determinants are $1, -1, -2, 3$. *Note that over every field the first determinant is non-zero. This shows that the matrix has rank* 2. *Note also that over the field* $\mathbb{Z}/2\mathbb{Z}$ *the before last determinant is zero, and over* $\mathbb{Z}/3\mathbb{Z}$ *the last determinant is zero. This shows that if some* $k \times k$ *sub-determinant is zero this need not hold for other* $k \times k$ *sub-determinants.*

4. Give two more proofs of the following theorem

Theorem 1. *Let A be an $m \times n$ matrix. Then*

$$\operatorname{rank}_c(A) = \operatorname{rank}_r(A).$$

(1) By applying Exercise 3 and properties of determinants.

(2) By completing the following argument. Write

$$A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix}.$$

If $r = \operatorname{rank}_r(A)$ then there exist vectors $S_1, \dots, S_r$ in $\mathbb{F}^n$ such that for suitable k_{ij}

$$A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix} = \begin{pmatrix} k_{11} & \dots & k_{1r} \\ \vdots & & \vdots \\ k_{m1} & \dots & k_{mr} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_r \end{pmatrix}.$$

(Explain why!). Now interpret each matrix as a linear transformation and see what you can say about the column rank!