Algebra II 189-251B **Instructor: Dr. E. Goren.**

Assignment 4

To be submitted by February 9, 12:00

In this assignment V and W are vector spaces of dimension n and m respectively, over a field $\mathbb{F}$.

1. Let $\mathcal{B}$ be a basis for V and $\mathcal{C}$ a basis for W. Prove that for any $m \times n$ matrix A with entries in $\mathbb{F}$ there exists a unique linear transformation $T : V \longrightarrow W$ such that ${}_{\mathcal{C}}[T]_{\mathcal{B}} = A$.

2. Deduce from the theorems on determinants the following:
   1. If a column is zero, the determinant is zero.
   2. $\det(A) = \det(A^t)$, where A^t is the transposed matrix.
   3. If a row is zero, the determinant is zero.
   4. Let A be a matrix in "block form":
   $$A = \begin{pmatrix} A_1 & 0 & & \star \\ 0 & A_2 & & \\ & & \ddots & \\ 0 & & 0 & A_k \end{pmatrix}.$$
   Here each A_i is a square matrix say of size r_i, and A_2 starts at the $r_1 + 1$ column and $r_1 + 1$ row, etc. Prove that
   $$\det(A) = \det(A_1)\det(A_2)\cdots\det(A_k).$$
   Conclude that the determinant of a triangular matrix is given by
   $$\det \begin{pmatrix} a_{11} & & & \star \\ 0 & a_{22} & & \\ & & \ddots & \\ 0 & & 0 & a_{kk} \end{pmatrix} = a_{11}a_{22}\cdots a_{kk}.$$
   (Here each a_{ii} is a scalar).

3. Calculate the following series of determinants.
   1. $\det\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\det\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \cdots$
   2. $\det\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, $\det\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\det\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $\det\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \cdots$
   3. $\det\left(\begin{smallmatrix} x & -a_2 \\ 1 & x+a_1 \end{smallmatrix}\right)$, $\det\begin{pmatrix} x & 0 & a_3 \\ 1 & x & -a_2 \\ 0 & 1 & x+a_1 \end{pmatrix}$, $\det\begin{pmatrix} x & 0 & 0 & -a_4 \\ 1 & x & 0 & a_3 \\ 0 & 1 & x & -a_2 \\ 0 & 0 & 1 & x+a_1 \end{pmatrix}, \cdots$

4. Prove the following formula (the *Vandermonde determinant*):
$$\det\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots x_1^{n-1} \\ \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots x_n^{n-1} \end{pmatrix} = \prod_{i>j}(x_i - x_j)$$

For example, for $n = 2, 3$ we have
$$\det\left(\begin{smallmatrix} 1 & x_1 \\ 1 & x_2 \end{smallmatrix}\right) = (x_2 - x_1), \quad \det\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$