Algebra II 189-251B **Instructor: Dr. E. Goren.**

Assignment 2

To be submitted by January 26, 12:00

1. Let V be the set of all sequences of complex numbers $(a_0, a_1, a_2, \dots)$ satisfying

$$a_n = a_{n-1} + a_{n-2}, \quad \forall n \geq 2.$$

Show that V has a natural structure of a vector space over $\mathbb{C}$. Find its dimension and a basis.

2. Let V be an n-dimensional vector space over a field $\mathbb{F}$. Let $T = \{t_1, \dots, t_m\} \subset V$ be a linearly independent set. Let $W = \mathrm{Span}(T)$. Prove:

$$\dim(W) = m.$$

3. Let W be a subspace of a vector space V of dimension n. Let $\{t_1, \dots, t_m\}$ be a basis for W. Prove that there exist vectors $\{t_{m+1}, \dots, t_n\}$ in V such that $\{t_1, \dots, t_m, t_{m+1}, \dots, t_n\}$ is a basis for V.

4. Let V_1, V_2 be finite dimensional vector spaces over a field $\mathbb{F}$. Prove that

$$\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2).$$

5. Let $\mathbb{F}$ be a field and $a_{11}, \dots, a_{1n}$ be scalars in $\mathbb{F}$. Let W_1 be the set of solutions to the linear equation $\sum_{j=1}^{n} a_{1j}x_j = 0$, namely,

$$W_1 = \{(x_1, \dots, x_n) \in \mathbb{F}^n : a_{11}x_1 + \cdots + a_{1n}x_n = 0\}.$$

Note that W_1 is a subspace of $\mathbb{F}^n$. Prove:

(1) $\dim(W_1) \geq n - 1$.

(2) Assume we are given scalars a_{ij}, $1 \leq i \leq k$, $1 \leq j \leq n$. Assume also that $k < n$. Let W be the solution set of the system of equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= 0 \\ \vdots \quad & \quad \vdots \\ a_{k1}x_1 + \cdots + a_{kn}x_n &= 0. \end{aligned}$$

Prove that $\dim(W) \geq n - k$. (Suggestion: Argue by induction on k. Use $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$).

6. Let $\mathcal{B} = \{(1,1), (1,5)\}$ and $\mathcal{C} = \{(2,1), (1,-1)\}$ be bases of $\mathbb{R}^2$. Find the change of bases matrices ${}_{\mathcal{B}}M_{\mathcal{C}}$ and ${}_{\mathcal{C}}M_{\mathcal{B}}$ between the bases $\mathcal{B}$ and $\mathcal{C}$. Let $v = \left(\begin{smallmatrix} 8 \\ 28 \end{smallmatrix}\right)$ with respect to the standard basis. Find $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{C}}$.

7. Let $\mathbb{F}$ be a finite field with q elements.
   (1) Show that the kernel of the ring homomorphism
   $$\mathbb{Z} \longrightarrow \mathbb{F}$$
   defined by $n \mapsto n \cdot 1 = 1 + \cdots + 1$ (n times) is of the form $p\mathbb{Z}$ for some prime p. Conclude that we may assume that $\mathbb{F} \supseteq \mathbb{Z}/p\mathbb{Z}$ for some prime p.
   (2) Prove that $\mathbb{F}$ is a vector space of finite dimension over $\mathbb{Z}/p\mathbb{Z}$ and if this dimension is n then $\mathbb{F}$ has p^n elements[1].

[1] Note: at this point you've proven that every finite field has cardinality p^n for some prime p.