Math 235 (Fall 2009): Assignment 2 Solutions

1: Let us prove by induction that $f(0,n) = \frac{n(n+1)}{2}$. It is clearly true in the case $n = 0$.

Assume it holds for n. Consider the diagonals in the second list (e.g. the diagonal $6, 7, 8, 9$). Notice that, by construction, the first diagonal (0) has 1 element, the second $(1, 2)$ has 2 elements, the third $(3, 4, 5)$ has 3 elements and, in general, the n-th diagonal has n elements. $f(0, n)$ is the number at the top part of the $n+1$-th diagonal and $f(0, n+1)$ is the number at the top part of the $n+2$-th diagonal. Hence, $f(0, n+1) = f(0, n) + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$ and this finishes the induction.

By construction it is easy to see that $f(1, n) = f(0, n+1)+1$, $f(2, n) = f(1, n+1)+1 = f(0, n+2)+1+1 = f(0, n+2)+2$, $f(3, n) = f(2, n+1)+1 = f(1, n+2)+2 = f(0, n+3) + 3$ and, in general, $f(m, n) = f(0, n+m) + m = \frac{(n+m)(n+m+1)}{2} + m$.

Now we shall prove that f is really bijective. Notice that, by definition of f, it satisfies $f(m, n) = f(0, n+m) + m$ and $f(0, n+m+1) = f(n+m, 0) + 1$. Hence, $f(0, n+m) < f(1, n+m-1) < f(2, n+m-2) < ... < f(m, n) < ... < f(n+m, 0) < f(0, n+m+1)$. Not only this but each term in the previous inequalities are 1 plus the previous term. Hence, it is clear that f is injective and surjective.

2: By the definition, $|A_1| = |A_2|$ means that there is a bijective function $f : A_1 \to A_2$. Since $|B_1| = |B_2|$, we also have a bijective function $g : B_1 \to B_2$.

We want to show that $|A_1 \times B_1| = |A_2 \times B_2|$, that is, we want to prove that there is a bijective function $h : A_1 \times B_1 \to A_2 \times B_2$.

Define the function $h(a_1, b_1) := (f(a_1), g(b_1))$. Let us prove h is bijective.

h is injective: In fact, if $h(a_1, b_1) = h(a_1', b_1')$, then $(f(a_1), g(b_1)) = (f(a_1'), g(b_1'))$, i.e., $f(a_1) = f(a_1')$ and $g(b_1) = g(b_1')$. But since f and g are injective functions, $a_1 = a_1'$ and $b_1 = b_1'$, that is, $(a_1, b_1) = (a_1', b_1')$.

h is surjective: Let $(a_2, b_2) \in A_2 \times B_2$, i.e., $a_2 \in A_2$ and $b_2 \in B_2$. Since f and g are surjective functions, there is $a_1 \in A_1$ and $b_1 \in B_1$ such that $f(a_1) = a_2$ and $g(b_1) = b_2$. Then, by definition of h, we have $h(a_1, b_1) = (a_2, b_2)$.

3: By Cantor–Bernstein–Schroeder theorem, it is enough to show two injective functions $f : \mathbb{N} \to \mathbb{Q}$, $g : \mathbb{Q} \to \mathbb{N}$.

Define $f(n) := n$. Clearly f is injective.

Before defining g we will define an injective function $h : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$. If $x \in \mathbb{Q}$, there are unique $a \in \mathbb{Z}$ and $b \in \mathbb{N}\backslash\{0\}$ such that $x = \frac{a}{b}$ (why?). Define $h(x) := (a, b)$. It is easy to see that h is injective.

By proposition 4.0.5 (in the notes), we have that $|\mathbb{Z}| = |\mathbb{N}|$. Hence, by exercise 2, we have $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N} \times \mathbb{N}|$. Now, using proposition 4.0.6 we conclude that $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}|$. Hence there is a bijective (hence, injective) function $j : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$.

Now define $g := j \circ h : \mathbb{Q} \to \mathbb{N}$. Since both h and j are injective, g is injective.

4: There is a missing hypothesis in this exercise: $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$. Now, consider the case $A_1 = A_2 = B_1 = \mathbb{Z}$, $B_2 = \mathbb{N}$. Clearly $|A_1| = |B_1|$. By proposition 4.0.5 in the notes, $|A_2| = |B_2|$. But since $A_1\backslash B_1 = \emptyset$ and $A_2\backslash B_2 \neq \emptyset$, $|A_1\backslash B_1| \neq |A_2\backslash B_2|$ (why?).

5: It is easy to see that $f : A \to 2^A$ defined by $f(a) := \{a\}$ is an injection.

Now suppose there is a bijection $g : 2^A \to A$. Define $U := \{a \in A : a \notin g(a)\}$. Since g is a bijection and $U \in 2^A$, $U = g(a_0)$ for some $a_0 \in A$.

Is a_0 in U? If $a_0 \in U$, $a_0 \notin g(a_0) = U$ (by the definition of U). If $a_0 \notin U$, since $U = g(a_0)$, we get that $a_0 \notin g(a_0)$ and, by definition of U, $a_0 \in U$.

Therefore, $a_0 \in U \Leftrightarrow a_0 \notin U$. This is a contradiction. Therefore, such a g does not exist.

6: *Fact*: Let $z \in \mathbb{C}$; then $|z| = 0$ if and only if $z = 0$. In fact, let $z = a + bi$. Then $|z| = 0 \Leftrightarrow |z|^2 = 0 \Leftrightarrow a^2 + b^2 = 0 \Leftrightarrow (a, b) = (0, 0) \Leftrightarrow z = 0$.

Now, since $|z_1||z_2| = |z_1 z_2| = 0$, then $|z_1| = 0$ or $|z_2| = 0$. In the first case we get, by the fact, $z_1 = 0$. In the second case, $z_2 = 0$.

7: We know that all the roots of the equation $ax^2 + bx + c = 0$ are given by $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Hence, the equation has only one solution if and only if $b^2 - 4ac = 0$, i.e., $b^2 = 4ac$.

Now, $x^2 + (1 + 6i)x + 1 = z$ has a unique solution if and only if $x^2 + (1 + 6i)x + (1 - z) = 0$ has a unique solution. And this happens when $(1 + 6i)^2 = 4(1 - z)$, i.e., $z = -\frac{(1+6i)^2}{4} + 1 = \frac{35-12i}{4} + 1 = \frac{39-12i}{4}$.

8: Let $z = x+bi$. Then $\Re(z)^2 = Im(z)^2$ if and only if $x^2 = y^2$, i.e., $(x+y)(x-y) = x^2 - y^2 = 0$. Moreover, $|z| \leq 1 \Leftrightarrow |z|^2 \leq 1 \Leftrightarrow x^2 + y^2 \leq 1$.Hence, on the complex plane, the points z such that $\Re(z)^2 = Im(z)^2$ and $|z| \leq 1$ form the segments (that lie inside the closed unit disc) of two perpendicular lines that meet at the point 0 (one line passes through $1 + i$ and the other passes through $1 - i$).